VECTORS

CONTENTS

MEQ VECTOR OBJECTIVES (8% of year)

(refer to pages 26 – 27 and 40 – 41 in curriculum guide)

Students will be introduced to two aspects of the theory of vectors: vectors represented by arrows in a plane or as ordered pairs of real numbers in the Cartesian plane. They will study vector addition, the multiplication of a vector by a scalar and the scalar product of two vectors. Students will also be able to demonstrate propositions using vectors. Finally, the students will be given problems related to what they have just learned about vectors, as well as real-life problems that involve applying these principles. (etc.)

Intermediate Objectives

- To perform the following operations on vectors: addition, multiplication by a scalar and the scalar multiplication of two vectors.
- To determine the properties of operations on vectors.
- To define the concept of the basis of a vector space.
- To prove propositions related to vectors.
- To prove propositions using vectors.
- To justify statements used in solving a problem.

A list of propositions can be found on pages 40 – 41 of the curriculum guide. These are repeated later in this document.

Please note that throughout this document, there are sample exercises suggested from the student text, Mathematical Reflections, Book 2, Reflection 6 (CEC). These exercises are meant to be samples only. Please refer to the MAPCO 536 Reflections guide for this chapter for other appropriate exercises.

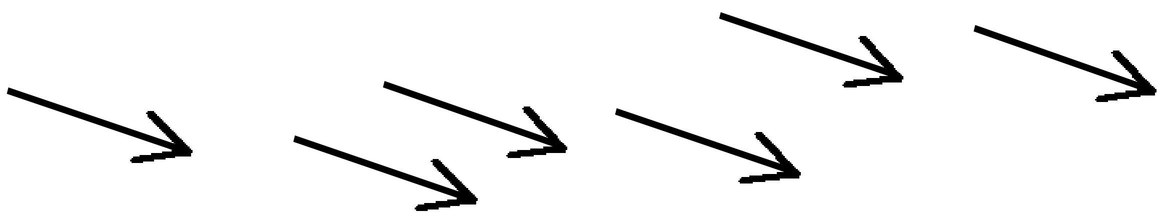
WHAT IS A VECTOR?

A directed line segment.
A line segment which has magnitude and direction.

A vector can represent displacement, velocity, acceleration and force.

A vector can be represented by a directed line segment (arrow).

If a vector is moved by parallel displacement, its direction and magnitude remain unchanged.
These arrows below all represent the same vector.



Exercise: page 123 part a.

Please note that if you are referring to one of the French textbooks for vector concepts, you will notice that there are three criteria for vectors: “longeur” (magnitude), “sens” (direction) and “direction” (orientation). “Sens” refers to which way the head of the vector is pointing, while “direction” refers to the slope of the vector. Since our English text refers only to magnitude and direction, this document follows this scheme.

WHAT IS A SCALAR?

A real number that can be represented on a scale or number line.
i.e. a scalar has magnitude, but no direction.

EXAMPLES:

VECTORS	SCALARS
50 km, N 30 degrees E 6 km 	34 degrees Celsius 89 kg 100 km/h 188 cm^3 7 hours

Exercise: page 129 # 1

Also from MAPCO binder June 1999:
For each of the following, determine which situations involve scalars and which involve vectors:

a. pulling a rope at an angle of 45 degrees with all my might
b. the outside thermometer reading of 23^oC
c. pushing a water wheel with a force of 15 N
d. the space station moving from West to East at 27 000 km/h
e. a polygon of perimeter 3 cm which represents the area of a field of 30 m^2
f. a line segment, AB, 5 cm long, at an angle of 45^o to the x-axis, representing a boat sailing towards the Northeast at 25 knots
g. a cargo container being loaded onto a ship
h. the amount of gas in a tank which is almost empty
i. slowly backing out of a parking spot at the mall
j. having enough water in the fish tank at home

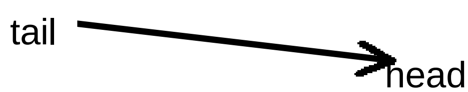
Answers: Scalars b e h j Vectors a c d f g i

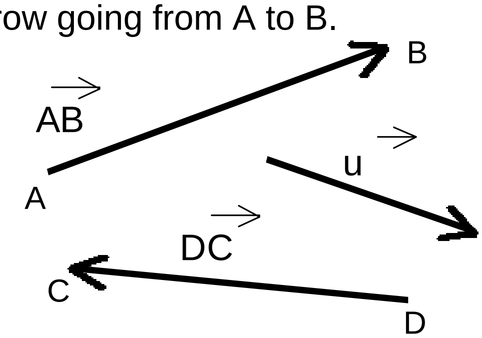
VECTORS FROM TWO PERSPECTIVES

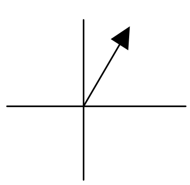
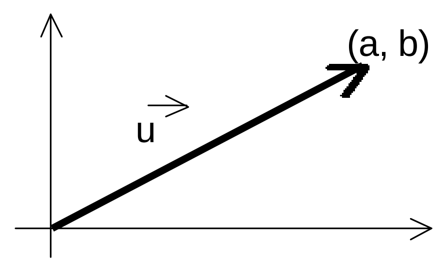
GEOMETRIC and **ALGEBRAIC**

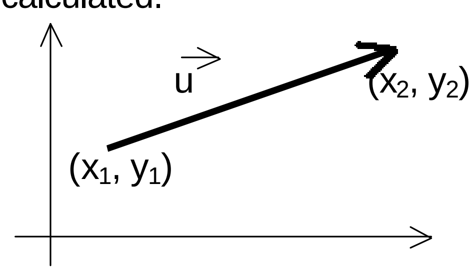
1. REPRESENTATION OF A VECTOR AND VECTOR NOTATION

Vectors do not have position. The displacement from one point to another doesn't tell us where we started or ended, only how we moved.

<table>
<tr><th>GEOMETRIC APPROACH</th><th>ALGEBRAIC APPROACH</th></tr>
<tr><td>
Every vector has a tail (initial point) and a head (terminal point).


Two points A and B, taken in this order, represent a vector $\overrightarrow{AB}$. A vector $\overrightarrow{AB}$ is represented by an arrow going from A to B.


Vector directions can also be identified by reference to compass headings.
N 30° E

Page 129, question 2 and page 130 question 3.
</td><td>
Vectors are represented by ordered pairs in a Cartesian plane. In the ordered pair (a, b) the x coordinate represents the horizontal component of the vector, and the y coordinate represents the vertical component of the vector.


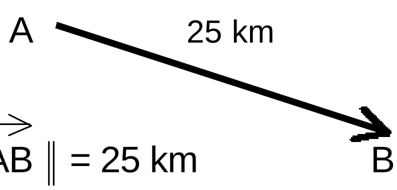
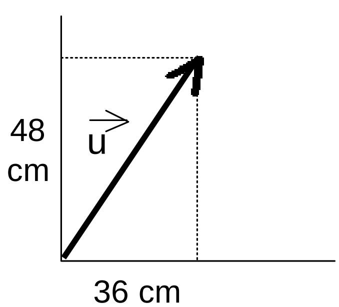
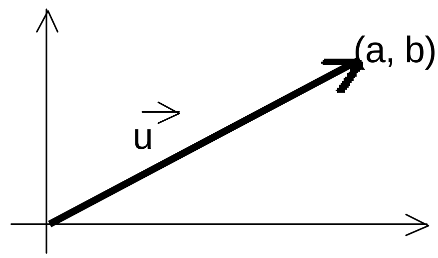
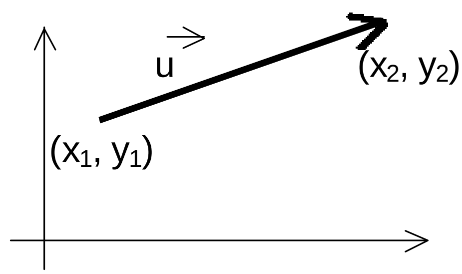
$\vec{u} = (a, b)$
When a vector's tail does not originate at (0, 0) the horizontal and vertical components must be calculated.


$\vec{u} = (x_2 - x_1, y_2 - y_1) = (a, b)$
Page 130, # 7, 8. Page 131 #14
</td></tr>
</table>

2. MAGNITUDE (NORM) OF A VECTOR

The **magnitude** or **norm** of a vector is the length of the vector.

The magnitude of vector $\vec{u}$ is written as $\|\vec{u}\|$.

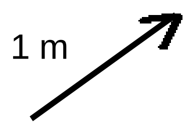
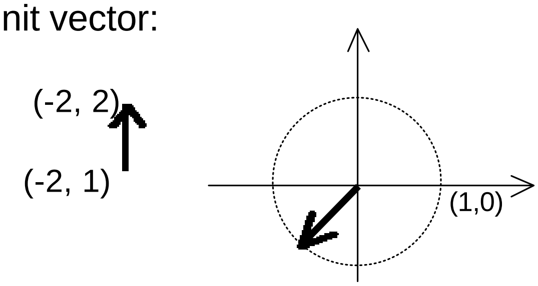
GEOMETRIC APPROACH	ALGEBRAIC APPROACH
The length of a vector can be measured if the drawing is to scale.  $\|\vec{AB}\| = 25$ km *Page 125, question f) and page 131 question 17.* The length can also be calculated using the Pythagorean Theorem, the Sine Law, or the Cosine Law, depending on the information given.  $\|\vec{u}\| = \sqrt{36^2 + 48^2}$ *Page 132, questions 18 a, b*	The length of a vector can be calculated by Pythagorean relationships.  $\|\vec{u}\| = \sqrt{a^2 + b^2}$  $\|\vec{u}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ *Page 131 questions 11 and 12*.

3. SPECIAL VECTORS: UNIT VECTOR, ZERO VECTOR

Definitions:

A **unit vector** has a magnitude of 1 unit. $\|\vec{u}\| = 1$

A **zero vector** has no length, and can be considered to have any direction. $\|\vec{u}\| = 0$
It has the same beginning and ending point.

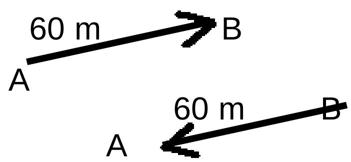
GEOMETRIC APPROACH	ALGEBRAIC APPROACH
Unit vector:  	Unit vector:  A unit vector with its tail at the origin will have its head on the standard unit circle.
Zero vector: $\vec{AA} = \vec{0}$	Zero vector: $\vec{0} = (0, 0)$

4. VECTOR RELATIONSHIPS

a) OPPOSITE VECTORS: **Opposite vectors** have the same magnitude but opposite directions They are parallel. Every vector has an opposite vector.

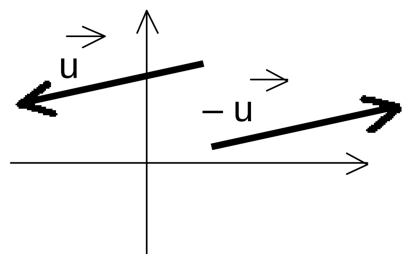
<table>
<tr><th>GEOMETRIC APPROACH</th><th>ALGEBRAIC APPROACH</th></tr>
<tr><td>Opposite vector:

$\overrightarrow{AB}$ $\qquad -\overrightarrow{AB} = \overrightarrow{BA}$



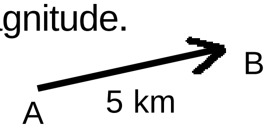
Page 134 question 1</td><td>Opposite vector:

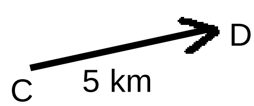
$\overrightarrow{u} = (a, b) \qquad -\overrightarrow{u} = (-a, -b)$



Page 135 question 9</td></tr>
</table>

b) EQUIVALENT VECTORS (also called EQUAL Vectors)
When arrow AB and arrow CD represent the same vector, we write: $\overrightarrow{AB} \uparrow \overrightarrow{CD}$

<table>
<tr><th>GEOMETRIC APPROACH</th><th>ALGEBRAIC APPROACH</th></tr>
<tr><td>Vectors which are equivalent to each other have the same direction and the same magnitude.


$\overrightarrow{AB} \uparrow \overrightarrow{CD}$


(quadrilateral ABDC will be a parallelogram)

Page 126, question h.</td><td>Vectors which are equivalent to each other have the same components.

Given vectors $\overrightarrow{u}$ and $\overrightarrow{v}$, such that

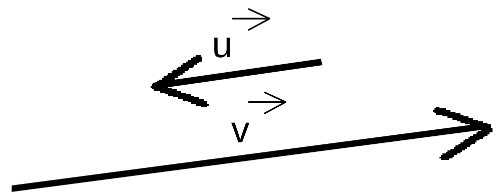
$\overrightarrow{u} = (a_1, b_1)$ and $\overrightarrow{v} = (a_2, b_2)$

$\overrightarrow{u} \uparrow \overrightarrow{v} \Leftrightarrow a_1 = a_2$ and $b_1 = b_2$

We can also write
$(a_1, b_1) \uparrow (a_2, b_2)$</td></tr>
</table>

Vectors can be equivalent, or equal. Arrows which represent the same vector are equipollent.

c) COLLINEAR VECTORS: When arrows representing two vectors are parallel, the vectors are **collinear**, or **linearly dependent**. The length of one vector will be a multiple of the length of the other vector.

GEOMETRIC APPROACH	ALGEBRAIC APPROACH
$\vec{u}$ and $\vec{v}$ are collinear $\Leftrightarrow \exists\, k \in \Re$ such that $\vec{v} = k\,\vec{u}$.  $\vec{v} = -2.5\,\vec{u}$	$\vec{u} = (a, b)$ and $\vec{v} = (c, d)$ are collinear $\Leftrightarrow ad - bc = 0$. For example: $\vec{v} = 2\,\vec{u}$ $\vec{u} = (3, 5)$ and $\vec{v} = (6, 10)$ $ad - bc = 3(10) - 5(6) = 0$

d) ORTHOGONAL VECTORS: Two vectors are **orthogonal** if and only if the arrows representing them are perpendicular to each other.

Exercises to review terms for these 4 relationships:
page 134, question 4 and page 136, question 10.

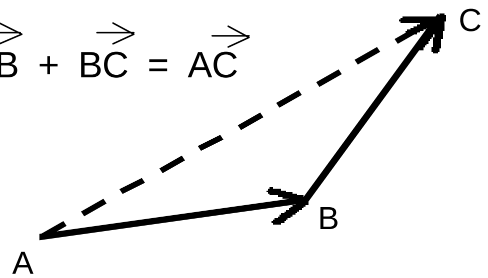
5. VECTOR ADDITION

Vector addition may represent the sum of forces, displacements, velocities or accelerations. When two vectors are added, the vector sum is called the **resultant vector**.

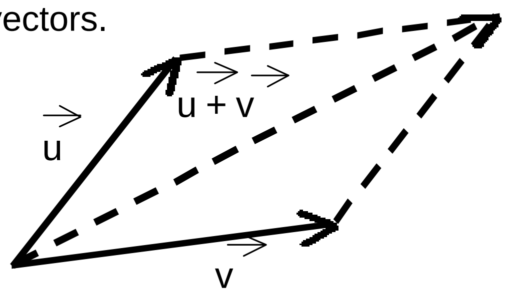
GEOMETRIC APPROACH

Michel Chasles (1793 – 1880) discovered a relation that exists for vector sums. This is useful when the vectors can be considered <u>sequentially</u>. The resultant vector is the vector from the initial starting point to the final ending point.

$\vec{AB} + \vec{BC} = \vec{AC}$



If two vectors are considered <u>from the same initial point</u>, the parallelogram law of vector addition can be used. Consider the two vectors as adjacent sides of a parallelogram, draw the other two sides, and the diagonal from the vertex of the two tails. This diagonal represents the sum of the two vectors.



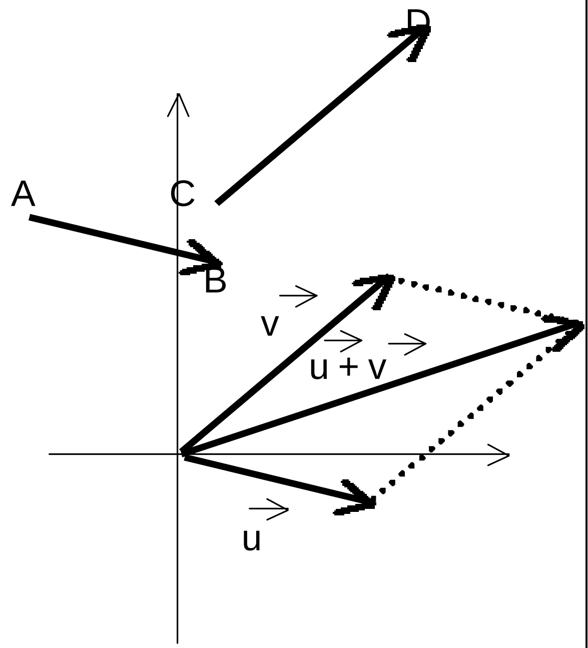
In either case, calculate the magnitude of the resultant using the **cosine law**. (Remember that the sum of the angles in a parallelogram is 360^o.)

Page 141 c, 148, # 14 and 17.

ALGEBRAIC APPROACH

Given $\vec{u} = (a, b)$ and $\vec{v} = (c, d)$

$\vec{u} + \vec{v} = (a + c, b + d)$



Example: $\vec{u} = \vec{AB}$ and $\vec{v} = \vec{CD}$ where A = (– 4, 6) and B = (1, 5) C = (1, 7) and D = (8, 12)

$\vec{u} = (1 – (– 4), 5 – 6) = (5, – 1)$

$\vec{v} = (8 – 1, 12 – 7) = (7, 5)$

$\vec{u} + \vec{v} = (5 + 7, –1 + 5) = (12, 4)$

Page 147, question 8 (draw on grid), Page 147, question 12; pages 149-150.

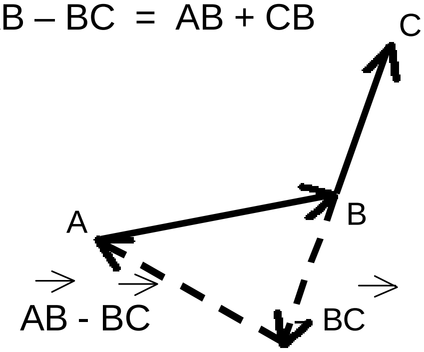
6. VECTOR SUBTRACTION

<table>
<tr><th>GEOMETRIC APPROACH</th><th>ALGEBRAIC APPROACH</th></tr>
<tr><td>

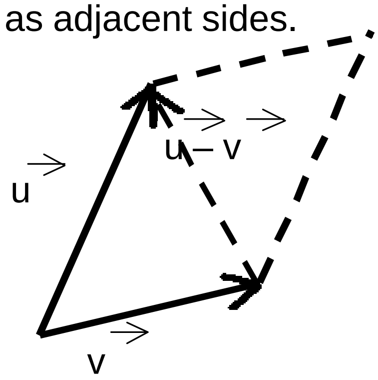
When two vectors are considered <u>sequentially</u>, the difference of the vectors is found by adding the opposite vector.

$\vec{AB} - \vec{BC} = \vec{AB} + (-\vec{BC})$

or, $\vec{AB} - \vec{BC} = \vec{AB} + \vec{CB}$



When two vectors are considered <u>from the same initial point</u>, subtraction of vectors can be viewed as the diagonal connecting the two heads in a parallelogram with the two initial vectors as adjacent sides.



(Note that the head of the difference vector points towards the head of the first vector.)

Page 141, k; 142 l, m; 143, n, o; 144, s; 146, 6 and 8; 147, 9

</td><td>

Given $\vec{u} = (a, b)$ and $\vec{v} = (c, d)$

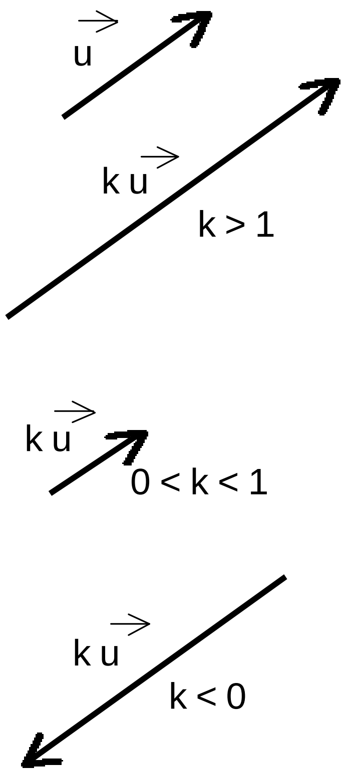
$\vec{u} - \vec{v} = (a - c, b - d)$

</td></tr>
</table>

7. MULTIPLICATION OF A VECTOR BY A SCALAR

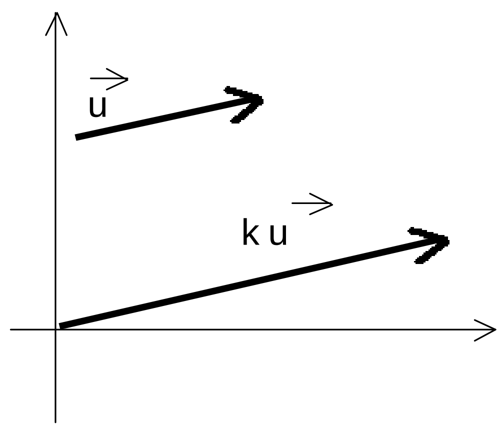
The multiplication of a vector by a positive scalar results in a vector having the same direction as the initial vector, with a length in proportion to the scalar (scale factor). If the scalar is negative, the direction of the resulting vector will be opposite to that of the original vector.

Example: $3\vec{u} = \vec{u} + \vec{u} + \vec{u}$

<table>
<tr><th>GEOMETRIC APPROACH</th><th>ALGEBRAIC APPROACH</th></tr>
<tr><td>


Page 154, questions 6 and 7.
Page 155, question 12 a.
</td><td>
Given
scalar k and vector $\vec{u} = (a, b)$

$k\vec{u} = k\,(a, b) = (ka, kb)$



Page 153, question 1 a, b and 2 a
Page 155, question 11 a and b.
</td></tr>
</table>

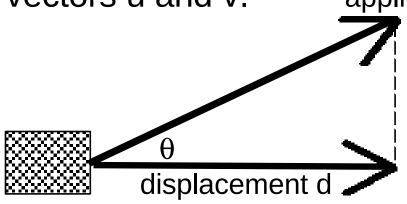
Note that $k\vec{u}$ is a vector, and $k\,\|\vec{u}\|$ is a scalar (magnitude only).

Note also that $\|k\vec{u}\| = |k| \times \|\vec{u}\|$

8. VECTOR SCALAR PRODUCT (or DOT PRODUCT)

The scalar product of two vectors is a scalar.
The scalar product is written as: $\vec{AB} \bullet \vec{CD}$ or $\vec{u} \bullet \vec{v}$

In physics, the scalar product gives the Work Done when a force is applied to an object and it displaces it a certain distance in a certain direction. This is represented by the formula W = Fd. (1 Newton metre = 1 joule)

GEOMETRIC APPROACH	ALGEBRAIC APPROACH
$\vec{u} \bullet \vec{v} = \lVert\vec{u}\rVert \lVert\vec{v}\rVert \cos\theta$ where θ is the angle between vectors $\vec{u}$ and $\vec{v}$.  *Page 160, question 4 a and b* *Page 161, question 12.*	$\vec{AB} \bullet \vec{CD} = (a, b) \bullet (c, d)$ $= ac + bd$ *Page 159, question 1* *Page 160, question 6*

When two vectors are perpendicular (orthogonal), the angle between them is 90 degrees, and the cosine of that angle is 0.

$$\text{Thus: } \vec{u} \perp \vec{v} \Leftrightarrow \vec{u} \bullet \vec{v} = 0$$

When $\theta = 0$ (vectors are collinear), $\cos\theta = 1$, (work is done efficiently).
When $\theta > 90^o$, $\cos\theta < 0$ and therefore $\vec{u} \bullet \vec{v} < 0$ (like taking a dog for a walk!)

Properties of scalar (dot) products:

Commutative: $\vec{u} \bullet \vec{v} = \vec{v} \bullet \vec{u}$

Associative: $(k\,\vec{u}) \bullet \vec{v} = k(\vec{u} \bullet \vec{v})$

Distributive: $\vec{w} \bullet (\vec{u} + \vec{v}) = (\vec{w} \bullet \vec{u}) + (\vec{w} \bullet \vec{v})$

9. PROPERTIES OF OPERATIONS ON VECTORS (see Topic 4)

Commutative properties of addition and multiplication:

$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{BC} + \overrightarrow{AB}$

$\vec{u} + \vec{v} = \vec{v} + \vec{u}$

$(k\,m)\ \vec{u} = (m\,k)\ \vec{u}$

Associative properties of addition and multiplication:

$(\overrightarrow{AB} + \overrightarrow{BC}) + \overrightarrow{CD} = \overrightarrow{AB} + (\overrightarrow{BC} + \overrightarrow{CD})$

$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

$k\,(m\,\vec{u}) = (k\,m)\,\vec{u}$

Distributive property of multiplication over addition or subtraction:

$k\,(\overrightarrow{AB} \pm \overrightarrow{CD}) = k\,\overrightarrow{AB} \pm k\,\overrightarrow{CD}$

$k\,(\vec{u} \pm \vec{v}) = k\,\vec{u} \pm k\,\vec{v}$

$(k \pm m)\ \vec{u} = k\,\vec{u} \pm m\,\vec{u}$

Identity Element for addition – the zero vector:

$\overrightarrow{AB} + \vec{0} = \overrightarrow{AB}$

$(x, y) + (0, 0) = (x, y)$

Identity Element for multiplication – the unit vector.

Additive Inverse – the opposite vector:

$\overrightarrow{AB} + (-\overrightarrow{AB}) = \vec{0}$

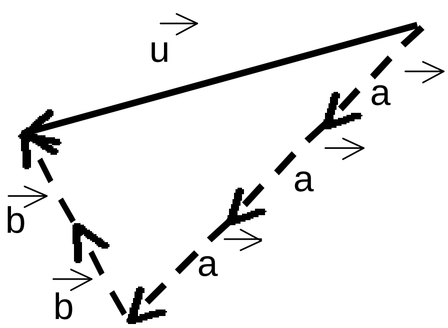
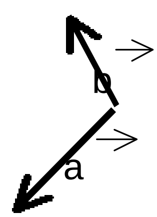
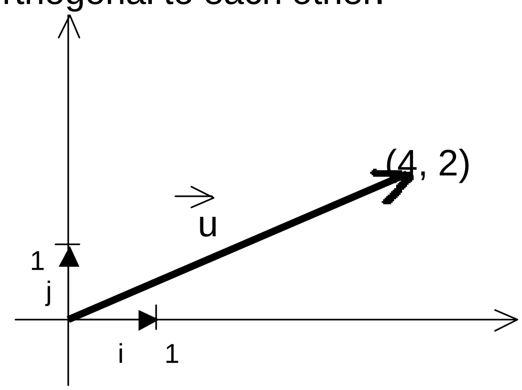
$\vec{u} + (-\vec{u}) = (a, b) + -(a, b) = (a, b) + (-a, -b) = (0, 0)$

Exercises pages 166 and 167.

10. BASIS OF A VECTOR SPACE

This topic relates to collinear vectors where $\vec{v} = k\,\vec{u}$.
It also relates to the addition of vectors, and their resultant.

Any vector can be written as a **linear combination** of two basis vectors, as long as the basis vectors aren't parallel to each other (are linearly independent).

GEOMETRIC APPROACH	ALGEBRAIC APPROACH
 $\vec{u} = 3\,\vec{a} + 2\,\vec{b}$  Vectors $\vec{a}$ and $\vec{b}$ represent basis vectors, and $\vec{u}$ is a vector represented by a linear combination of that vector base. *Page 176, question 3, 5*	The Cartesian plane has a vector basis formed by two unit vectors, i and j, which are parallel to the x and y axes respectively, and are orthogonal to each other.  $\vec{u} = 4\,\vec{i} + 2\,\vec{j}$ $\vec{i} = (1, 0)$ and $\vec{j} = (0, 1)$ so $\vec{u} = 4\,(1, 0) + 2\,(0, 1)$ All vectors can be expressed as a linear combination of $\vec{i}$ and $\vec{j}$. *Page 174, question f*

The Cartesian plane is an example of an **orthonormal** coordinate system with basis vectors which are unit vectors, orthogonal to each other.

PROPOSITIONS:

Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors in the plane and r and s, scalars.
Verify the following properties:

Page 180	1. $(r\vec{u} = \vec{0}) \Leftrightarrow (r = 0 \text{ or } \vec{u} = \vec{0})$
Page 182	2. if $\vec{u}$ and $\vec{v}$ are non-collinear vectors then $(r\vec{u} = s\vec{v}) \Leftrightarrow (r = s = 0)$
Page 179	3. $(\vec{w}$ is collinear with $\vec{u}) \Leftrightarrow (\exists\, r\ \varepsilon\ \Re\text{: } \vec{w} = r\vec{u})$
	4. $(\vec{u}$ and $\vec{v}$ are non-collinear$) \Leftrightarrow (\forall\, \vec{w}, \exists\, r\ \varepsilon\ \Re,\ \exists\, s\ \varepsilon\ \Re\text{: } \vec{w} = r\vec{u} + s\vec{v})$
Page 184	5. $(\vec{u} \perp \vec{v}) \Leftrightarrow (\vec{u} \bullet \vec{v} = 0)$

Using vectors, **prove** the following propositions:

Page 188 (top)	6. The diagonals of a parallelogram bisect each other. In addition, a quadrilateral whose diagonals bisect each other is a parallelogram.
Page 187	7. The line segment joining the midpoints of two sides of a triangle is parallel to the third side and its length is one-half the length of the third side.
Page 189 (top)	8. The midpoints of the sides of any quadrilateral are the vertices of a parallelogram.
Page 193 # 15	9. If O is a point in the plane, and if P, Q and R are the midpoints of the sides of triangle ABC, then $\vec{OP} + \vec{OQ} + \vec{OR} = \vec{OA} + \vec{OB} + \vec{OC}$
Page 190	10. In any parallelogram, if a vertex is joined to the midpoints of the non-adjacent sides, then the opposite diagonal is cut into three equal segments.
Page 193 # 16	11. The medians of a triangle are concurrent in a point that is two-thirds the distance from each vertex to the midpoint of the opposite side.
Page 189 (bottom)	12. An angle inscribed in a semicircle is a right angle.
Page 194 # 17	13. In any triangle, the square of the length of a side is equal to the sum of the squares of the length of the other two sides, minus twice the product of the lengths of the other two sides and the cosine of the contained angle. (The Cosine Law)
Page 194 # 18	14. The diagonals of a rhombus are perpendicular.
	15. If the midpoints of the sides of an equilateral triangle are joined, an equilateral triangle is formed, each side of which measures half that of the original triangle.

(translated from Mathématiques 2000)

Helps in proving propositions using vectors:

1. To prove that quadrilateral ABCD is a parallelogram, you can prove that:

$$\overrightarrow{AB} = \overrightarrow{DC} \text{ or that } \overrightarrow{AD} = \overrightarrow{BC}.$$

2. To prove that line segments d and d' are parallel, you can prove that:
   the vectors of d and d' are collinear;
   or that the "normalized" vectors of d and d' are collinear

3. To prove that the line segments d and d' are perpendicular, you can prove that:
   the vectors of d and d' are orthogonal;
   or that the normalized vectors of d and d' are orthogonal;
   or that the scalar product of the vectors of d and d' is zero.

4. To prove that M is the middle of segment AB you can prove that:

   the vectors $\overrightarrow{AM}$ and $\overrightarrow{MB}$ are equal;

   or that the vectors $\overrightarrow{MA}$ and $\overrightarrow{MB}$ are opposite.

In all vector proofs, remember Chasles' Relation to simplify vector sums.

See also page 178 in Reflections for additional strategies.

WEBSITES TO VISIT (ON VECTORS)

www.forum.swarthmore.edu/~klotz/Vectors/vectors.html

http://www.physics.ohio-state.edu/~hughes/physics131/L4_vectors_131/sld001.htm

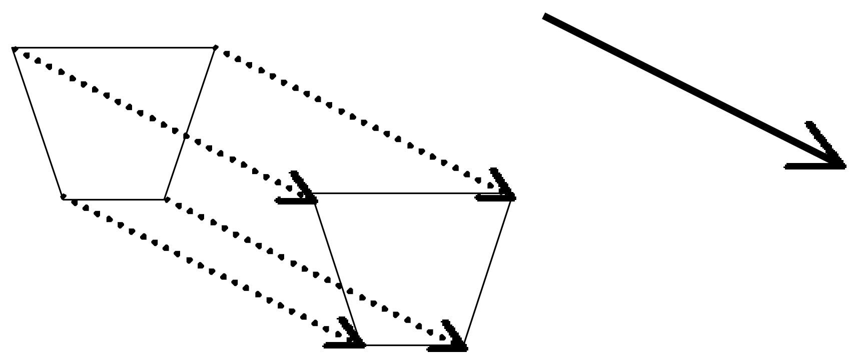
http://www.netcomuk.co.uk/~jenolive/homevec.html

http://webug.physics.uiuc.edu/courses/phys100/fall97/discussion_topics/Vectors/index.html

WHAT PRIOR CONCEPTS HAVE STUDENTS ENCOUNTERED?

MATH 116 – Translations in the plane

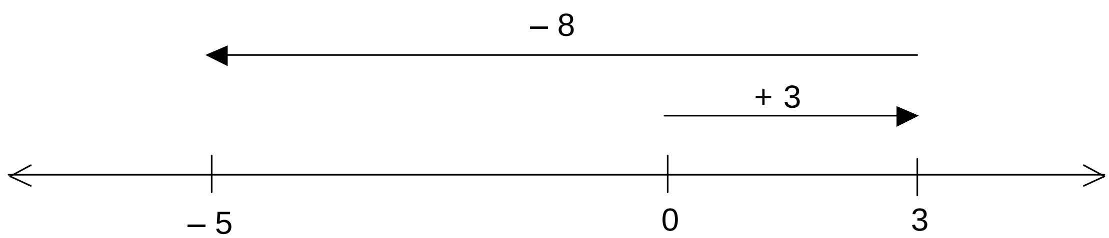
Students are given a polygon as an "object", and a translation arrow which shows direction and distance. They must translate the polygon vertex by vertex, by drawing lines parallel to the translation arrow, and measuring off the length of the arrow. This gives them the "image" of the polygon under the given translation.



MATH 116 – Number Line for Integer Operations

Some textbooks explain integer addition and subtraction with the use of directed line segments.

+ 3 + (– 8)



MATH 216 – TRANSFORMATIONS IN THE COORDINATE PLANE

Students are given a polygon on a coordinate plane, and a “mapping rule” for translations and dilatations. They then operate on the coordinates of the vertices of the polygon according to the mapping rule in order to construct the image of the polygon under the transformation.

$t_{(-2,4)} : (x,y) \mapsto (x - 2, y + 4)$

Example: coordinates of vertex A are (5, 2)
Coordinates of vertex A’ under the translation are (5 – 2, 2 + 4) = (3, 6)

$h_{(0,3)}: (x,y) \mapsto (3x, 3y)$

Example: coordinates of vertex P are (2, – 3)
Coordinates of vertex P’ under the dilatation are (3 x 2, 3 x – 3) = (6, – 9)

MATH 314 – COMPOSITIONS OF TRANSFORMATIONS

Students are asked to carry out two consecutive translations of an object, then to identify a single translation that would have the same effect as the composite of the two translations. They are also asked for the translation that would take the image back to the position of the object, or the inverse translation.

Given $t_1: (x,y) \mapsto (x + 2, x - 5)$ and $t_2 : (x,y) \mapsto (x, y + 2)$
Construct: $t_2 \mathbf{N} t_1$

Example: coordinates of vertex M are (– 3, 1)

Under t_1 the coordinates of M’ are (– 1, – 4)

Under t_2 the coordinates of M’’ are (– 1, – 2)

The composite translation $t_2 \mathbf{N} t_1$ is $(x, y) \mapsto (x + 2, y - 3)$

The composite inverse transformation $(t_2 \mathbf{N} t_1)^{-1}$ is $(x, y) \mapsto (x - 2, y + 3)$

MEQ 1999 Exam Questions On Vector Concepts

Q1.

Given the following information:

$\vec{a}$ and $\vec{b}$ are nonzero vectors in the plane
$\vec{a} \neq \vec{b}$
k is a scalar not equal to zero
$k \neq 1$

Which of the following statements is true?

A) $k\left(\vec{a} \bullet \vec{b}\right) = k\vec{a} \bullet k\vec{b}$

B) $k\left(\vec{a} + \vec{b}\right) = k\vec{a} + k\vec{b}$

C) If $\vec{a} \bullet \vec{b} = 0$, then $\vec{a}$ and $\vec{b}$ are collinear.

D) If $\vec{a} = k\vec{b}$, then $\vec{a}$ and $\vec{b}$ are noncollinear.

Q2.

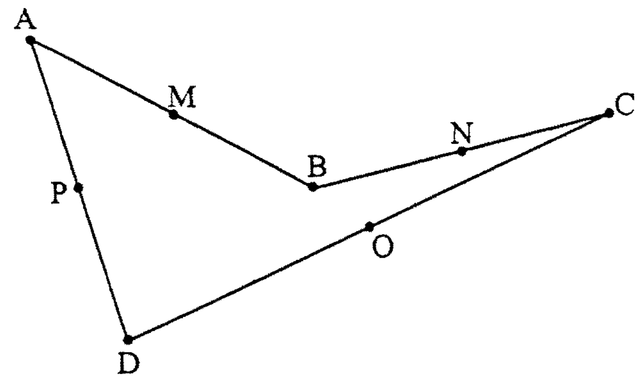
Given vectors u and v.

$\vec{u} = \overrightarrow{AB}$ where A(–2, 3) and B(6, 7)
$\vec{v} = (4, -4)$

What is the scalar product of vectors u and v?

Q3.

In quadrilateral ABCD illustrated below, points M, N, O and P are the midpoints of segments AB, BC, CD and DA respectively.



Using the above figure, prove the following proposition: "The midpoints of the sides of any quadrilateral form the vertices of a parallelogram."

Show all your work.