Linear Combinations and Span

Given two vectors $\mathbf{v}$ and $\mathbf{w}$, a **linear combination** of $\mathbf{v}$ and $\mathbf{w}$ is any vector of the form

$$a\mathbf{v} + b\mathbf{w}$$

where a and b are scalars. For example, the vector $(6, 8, 10)$ is a linear combination of the vectors $(1, 1, 1)$ and $(1, 2, 3)$, since

$$\begin{bmatrix} 6 \\ 8 \\ 10 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

For example, a linear combination of three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ would have the form $a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$, where a, b, and c are scalars.

More generally, a **linear combination** of n vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is any vector of the form

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$$

where $a_1, a_2, \ldots, a_n$ are scalars. For $n = 2$, this reduces to the definition for two vectors given above.

It is all right if some of the scalars in a linear combination are either zero or negative. For example, if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors, then

$$2\mathbf{u} - 3\mathbf{v} + 4\mathbf{w}, \qquad 3\mathbf{u} + 5\mathbf{w}, \qquad \mathbf{v} + \mathbf{w}, \qquad \mathbf{w} - \mathbf{u}, \qquad \text{and} \qquad 5\mathbf{v}$$

are some possible linear combinations of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

We will sometimes want to discuss linear combinations of a single vector. If $\mathbf{v}$ is a vector, a linear combination of just $\mathbf{v}$ is the same thing as a scalar multiple of $\mathbf{v}$:

$$a\mathbf{v}.$$

Thus $(3, 12, 6)$ is a linear combination of $(1, 4, 2)$, since $(3, 12, 6) = 3(1, 4, 2)$.

Expressing a Vector as a Linear Combination

Sometimes you want to express one vector as a linear combination of others. For example, can we express the vector $(8, 3, 3)$ as a linear combination of $(1, 1, 1)$ and $(1, 0, 0)$? A moment's thought reveals the answer:

$$\begin{bmatrix} 8 \\ 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

For more complicated examples, you can express one vector as a linear combination of others by solving a system of linear equations.

EXAMPLE 1 Express the vector $(9, 6)$ as a linear combination of the vectors $(1, 2)$ and $(1, -4)$.

SOLUTION We are looking for scalars x_1 and x_2 so that

$$x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \end{bmatrix}.$$

We can write this equation as a system of linear equations:

$$\begin{aligned} x_1 + x_2 &= 9 \\ 2x_1 - 4x_2 &= 6 \end{aligned}$$

Solving gives $x_1 = 7$ and $x_2 = 2$. Thus

$$\begin{bmatrix} 9 \\ 6 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -4 \end{bmatrix}. \qquad \blacksquare$$

EXAMPLE 2 Determine whether the vector $(2, 1, 3)$ is a linear combination of the vectors $(1, 2, 3)$ and $(2, 3, 1)$.

SOLUTION We are looking for scalars x_1 and x_2 so that

$$x_1 \begin{bmatrix}1\\2\\3\end{bmatrix} + x_2 \begin{bmatrix}2\\3\\1\end{bmatrix} = \begin{bmatrix}2\\1\\3\end{bmatrix}.$$

We can write this equation as a system of linear equations:

$$\begin{aligned} x_1 + 2x_2 &= 2 \\ 2x_1 + 3x_2 &= 1 \\ 3x_1 + x_2 &= 3 \end{aligned}$$

which we can solve using row reduction:

$$\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 2 & 3 & 1 \\ 3 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & -1 & -3 \\ 0 & -5 & -3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 0 & -5 & -3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 12 \end{array}\right].$$

If we had found a solution for x_1 and x_2, it would have meant that $(2, 1, 3)$ *was* a linear combination of $(1, 2, 3)$ and $(2, 3, 1)$.

By this point, it has become clear that the system of linear equations has no solutions. We conclude that $(2, 1, 3)$ is not a linear combination of $(1, 2, 3)$ and $(2, 3, 1)$. ■

The Span of Vectors

The **span** of a collection of vectors is the set of all possible linear combinations of them. For example, the span of the vectors $(1, 5, 3)$ and $(2, 1, 7)$ is the set of all vectors of the form

$$s \begin{bmatrix}1\\5\\3\end{bmatrix} + t \begin{bmatrix}2\\1\\7\end{bmatrix}$$

as s and t range over all possible scalars.

EXAMPLE 3 Describe the span of the vectors $(1, 0, 0)$ and $(0, 1, 1)$.

SOLUTION A linear combination of the vectors $(1, 0, 0)$ and $(0, 1, 1)$ has the form

$$s \begin{bmatrix}1\\0\\0\end{bmatrix} + t \begin{bmatrix}0\\1\\1\end{bmatrix}$$

which simplifies to

$$\begin{bmatrix}s\\t\\t\end{bmatrix}$$

where s and t many be any real numbers. Hence, the span of the vectors $(1, 0, 0)$ and $(0, 1, 1)$ is the set of all vectors in $\mathbb{R}^3$ whose second and third entries are the same. ■

EXAMPLE 4 Describe the span of the vector $(1, 4)$.

SOLUTION Recall that a linear combination of $(1, 4)$ is just any scalar multiple of $(1, 4)$:

$$t \begin{bmatrix}1\\4\end{bmatrix} = \begin{bmatrix}t\\4t\end{bmatrix}.$$

As t ranges over all real numbers, this gives all possible vectors whose y-component is 4 times the x-component. Thus, the span of the vector $(1, 4)$ is the line $y = 4x$ in $\mathbb{R}^2$. ■

There is an exception to this rule. If $\mathbf{v}$ is the zero vector, then any multiple of $\mathbf{v}$ is again the zero vector, so the span of $\mathbf{v}$ is not a line.

In general, if $\mathbf{v}$ is any vector in $\mathbb{R}^n$, then the span of $\mathbf{v}$ is the line in $\mathbb{R}^n$ consisting of all scalar multiples of $\mathbf{v}$. That is, the span of $\mathbf{v}$ is the line in $\mathbb{R}^n$ that goes through the origin as well as the point $\mathbf{v}$.

Something similar happens if you take the span of two vectors: the result is usually a plane. In particular:

- If you take the span of two vectors in $\mathbb{R}^2$, the result is usually the entire plane $\mathbb{R}^2$.
- If you take the span of two vectors in $\mathbb{R}^3$, the result is usually a plane through the origin in 3-dimensional space.
- Similarly, if you take the span of two vectors in $\mathbb{R}^n$ (where $n > 3$), the result is usually a plane through the origin in n-dimensional space.

More precisely, if you take the span of two vectors $\mathbf{v}$ and $\mathbf{w}$, the result is the plane that goes through the origin as well as the points $\mathbf{v}$ and $\mathbf{w}$.

The word "usually" is important here. For example, if $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^3$ and $\mathbf{w} = (0,0,0)$, then the span of $\mathbf{v}$ and $\mathbf{w}$ will be the same as all the multiples of $\mathbf{v}$, which is just a line. More generally, if $\mathbf{w}$ is itself a multiple of $\mathbf{v}$, then every linear combination of $\mathbf{v}$ and $\mathbf{w}$ is again a multiple of $\mathbf{v}$, so the span of $\mathbf{v}$ and $\mathbf{w}$ is just a line. This makes sense geometrically: if $\mathbf{w}$ is a multiple of $\mathbf{v}$ then $\mathbf{v}$ and $\mathbf{w}$ line on the same line through the origin, and the span of $\mathbf{v}$ and $\mathbf{w}$ is this line.

Using this intuition, it's not hard to find vectors whose span is a given line or plane.

EXAMPLE 5 Find a vector in $\mathbb{R}^2$ whose span is the line $y = 2x$.

SOLUTION We just need any vector at all that lies on this line, other than the zero vector. For example, the span of the vector $(1,2)$ is the line $y = 2x$. ■

EXAMPLE 6 Find two vectors in $\mathbb{R}^3$ whose span is the plane $2x - 6y + 5z = 0$.

SOLUTION Again, any two vectors on this plane will work, as long as they are not multiples of each other. But how can we make up points on this plane?

It really does work to just make up points on the plane. For example, observe that $(3,1,0)$ and $(5,0,-2)$ both lie on the plane $2x - 6y + 5z = 0$. These are not multiples of one another, so their span is the given plane.

The simplest method is to choose values for two of the variables, and then solve for the third variable. That is, we think of the plane as

$$x = 3y - \frac{5}{2}z$$

where y and z are free variables. Setting $y = 1$ and $z = 0$ gives the vector $(3,1,0)$, while setting $y = 0$ and $z = 1$ gives the vector $(-5/2, 0, 1)$. Both of these vectors lie on the plane, and they are not multiples of one another, so their span is the entire plane. ■

EXAMPLE 7 Let P be the plane in $\mathbb{R}^3$ that goes through the origin as well as the points $(5,1,3)$ and $(2,-1,2)$. Does the point $(4,5,0)$ lie on P?

SOLUTION The plane P is just the span of the vectors $(5,1,3)$ and $(2,-1,2)$. Thus the point $(4,5,0)$ will lie on this plane if and only if it can be expressed as a linear combination of $(5,1,3)$ and $(2,-1,2)$. So we want to know whether the equation

$$x_1 \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}.$$

has any solutions for x_1 and x_2. To find out, we must solve the linear system

$$\begin{aligned} 5x_1 + 2x_2 &= 4 \\ x_1 - x_2 &= 5 \\ 3x_1 + 2x_2 &= 0 \end{aligned}$$

Solving yields $x_1 = 2$ and $x_2 = -3$, i.e.

$$2\begin{bmatrix}5\\1\\3\end{bmatrix} - 3\begin{bmatrix}2\\-1\\2\end{bmatrix} = \begin{bmatrix}4\\5\\0\end{bmatrix}.$$

Since $(4,5,0)$ is a linear combination of $(5,1,3)$ and $(2,-1,2)$, this point does lie on the plane P. ■

EXERCISES

1–8 ■ Express the vector $\mathbf{w}$ as a linear combination of the given vectors $\mathbf{v}_i$. (Note: You ought to be able to solve problems 1–8 in your head.)

1. $\mathbf{v}_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}0\\1\end{bmatrix}$, $\mathbf{w} = \begin{bmatrix}3\\5\end{bmatrix}$

2. $\mathbf{v}_1 = \begin{bmatrix}2\\0\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}0\\3\end{bmatrix}$, $\mathbf{w} = \begin{bmatrix}8\\9\end{bmatrix}$

3. $\mathbf{v}_1 = \begin{bmatrix}2\\1\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}1\\0\end{bmatrix}$, $\mathbf{w} = \begin{bmatrix}13\\4\end{bmatrix}$

4. $\mathbf{v}_1 = \begin{bmatrix}5\\7\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}1\\4\end{bmatrix}$ $\mathbf{w} = \begin{bmatrix}5\\20\end{bmatrix}$

5. $\mathbf{v}_1 = \begin{bmatrix}-2\\3\end{bmatrix}$, $\mathbf{w} = \begin{bmatrix}-8\\12\end{bmatrix}$

6. $\mathbf{v}_1 = \begin{bmatrix}2\\0\\5\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}0\\2\\0\end{bmatrix}$, $\mathbf{w} = \begin{bmatrix}4\\-6\\10\end{bmatrix}$

7. $\mathbf{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$, $\mathbf{w} = \begin{bmatrix}2\\3\\8\end{bmatrix}$

8. $\mathbf{v}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}0\\2\\0\end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix}0\\0\\3\end{bmatrix}$, $\mathbf{w} = \begin{bmatrix}5\\9\\2\end{bmatrix}$

9–12 ■ Use the method of Example 1 to express the vector $\mathbf{w}$ as a linear combination of the vectors $\mathbf{v}_i$.

9. $\mathbf{v}_1 = \begin{bmatrix}3\\7\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}6\\-5\end{bmatrix}$, $\mathbf{w} = \begin{bmatrix}21\\30\end{bmatrix}$

10. $\mathbf{v}_1 = \begin{bmatrix}1\\2\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}3\\4\end{bmatrix}$, $\mathbf{w} = \begin{bmatrix}-2\\1\end{bmatrix}$

11. $\mathbf{v}_1 = \begin{bmatrix}1\\3\\7\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}2\\7\\4\end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix}2\\6\\5\end{bmatrix}$, $\mathbf{w} = \begin{bmatrix}3\\7\\23\end{bmatrix}$

12. $\mathbf{v}_1 = \begin{bmatrix}1\\2\\1\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}4\\3\\2\end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix}8\\1\\2\end{bmatrix}$, $\mathbf{w} = \begin{bmatrix}-3\\4\\1\end{bmatrix}$

13–14 ■ Determine whether the vector $\mathbf{w}$ lies in the span of the vectors $\mathbf{v}_i$.

13. $\mathbf{v}_1 = \begin{bmatrix}1\\2\\5\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}3\\1\\4\end{bmatrix}$, $\mathbf{w} = \begin{bmatrix}14\\3\\15\end{bmatrix}$

14. $\mathbf{v}_1 = \begin{bmatrix}1\\3\\7\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}2\\1\\4\end{bmatrix}$, $\mathbf{w} = \begin{bmatrix}4\\7\\3\end{bmatrix}$

15. Let L be the line in $\mathbb{R}^3$ that goes through the points $(0,0,0)$ and $(1,4,2)$. Does the point $(3,12,5)$ lie on this line? Explain.

16. Let P be the plane in $\mathbb{R}^3$ that goes through the points $(0,0,0)$, $(1,2,0)$, and $(0,0,1)$. Does the point $(3,6,4)$ lie on this plane? Explain.

17–22 ■ Determine whether the span of the given vectors $\mathbf{v}_i$ is a single point, a line, or all of $\mathbb{R}^2$. If the span is a line, give the equation for the line.

17. $\mathbf{v}_1 = \begin{bmatrix}3\\6\end{bmatrix}$.

18. $\mathbf{v}_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}0\\1\end{bmatrix}$.

19. $\mathbf{v}_1 = \begin{bmatrix}0\\0\end{bmatrix}$.

20. $\mathbf{v}_1 = \begin{bmatrix}2\\1\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}-1\\5\end{bmatrix}$.

21. $\mathbf{v}_1 = \begin{bmatrix}2\\1\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}0\\0\end{bmatrix}$.

22. $\mathbf{v}_1 = \begin{bmatrix}1\\3\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}2\\6\end{bmatrix}$.

23. Find a vector $\mathbf{v}_1$ in $\mathbb{R}^2$ whose span is the line $2x + 3y = 0$.

24. Find two vectors $\mathbf{v}_1, \mathbf{v}_2$ in $\mathbb{R}^3$ whose span is the plane $x + 3y - 4z = 0$.

Answers

1. $\mathbf{w} = 3\mathbf{v}_1 + 5\mathbf{v}_2$ **2.** $\mathbf{w} = 4\mathbf{v}_1 + 3\mathbf{v}_2$ **3.** $\mathbf{w} = 4\mathbf{v}_1 + 5\mathbf{v}_2$ **4.** $\mathbf{w} = 0\mathbf{v}_1 + 5\mathbf{v}_2$ (or simply $\mathbf{w} = 5\mathbf{v}_2$)

5. $\mathbf{w} = 4\mathbf{v}_1$ **6.** $\mathbf{w} = 2\mathbf{v}_1 - 3\mathbf{v}_2$ **7.** $\mathbf{w} = 2\mathbf{v}_1 + 3\mathbf{v}_2 + 8\mathbf{v}_3$ **8.** $\mathbf{w} = 5\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3$

9. $\mathbf{w} = 5\mathbf{v}_1 + \mathbf{v}_2$ **10.** $\mathbf{w} = \frac{11}{2}\mathbf{v}_1 - \frac{5}{2}\mathbf{v}_2$ **11.** $\mathbf{w} = 3\mathbf{v}_1 - 2\mathbf{v}_2 + 2\mathbf{v}_3$

12. Many answers are possible. For example, $\mathbf{w} = 5\mathbf{v}_1 - 2\mathbf{v}_2 + 0\mathbf{v}_3$. **13.** Yes, since $\mathbf{w} = -\mathbf{v}_1 + 5\mathbf{v}_2$.

14. No, since $\mathbf{w}$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. **15.** No, since $(3,12,5)$ is not a scalar multiple of $(1,4,2)$.

16. Yes, since $(3,6,4)$ is a linear combination of $(1,2,0)$ and $(0,0,1)$. **17.** The span is the line $y = 2x$.

18. The span is all of $\mathbb{R}^2$ **19.** The span is the single point $(0,0)$. **20.** The span is all of $\mathbb{R}^2$.

21. The span is the line $y = \frac{1}{2}x$. **22.** The span is the line $y = 3x$.

23. Many answers are possible. For example, $\mathbf{v}_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ suffices.

24. Many answers are possible. For example, $\mathbf{v}_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$ suffice.