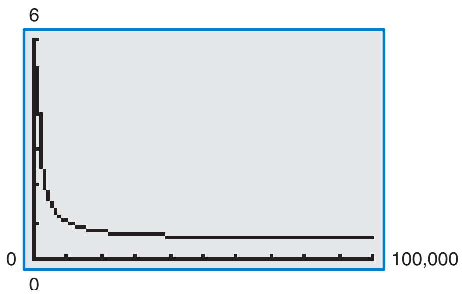
11 Limits and an Introduction to Calculus



Section 11.4, Example 3
Average Cost

11.1 Introduction to Limits
11.2 Techniques for Evaluating Limits
11.3 The Tangent Line Problem
11.4 Limits at Infinity and Limits of Sequences
11.5 The Area Problem

11.1 Introduction to Limits

The Limit Concept

The notion of a limit is a *fundamental* concept of calculus. In this chapter, you will learn how to evaluate limits and how they are used in the two basic problems of calculus: the tangent line problem and the area problem.

What you should learn

- Understand the limit concept.
- Use the definition of a limit to estimate limits.
- Determine whether limits of functions exist.
- Use properties of limits and direct substitution to evaluate limits.

Why you should learn it

The concept of a limit is useful in applications involving maximization. For instance, in Exercise 5 on page 757, the concept of a limit is used to verify the maximum volume of an open box.

Example 1 Finding a Rectangle of Maximum Area

You are given 24 inches of wire and are asked to form a rectangle whose area is as large as possible. What dimensions should the rectangle have?

Solution

Let w represent the width of the rectangle and let l represent the length of the rectangle. Because

$$2w + 2l = 24 \qquad \text{Perimeter is 24.}$$

it follows that

$$l = 12 - w$$

as shown in Figure 11.1. So, the area of the rectangle is

$$\begin{aligned} A &= lw && \text{Formula for area} \\ &= (12 - w)w && \text{Substitute } 12 - w \text{ for } l. \\ &= 12w - w^2. && \text{Simplify.} \end{aligned}$$

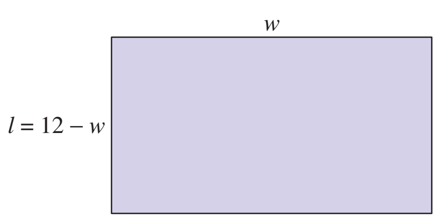


Figure 11.1

Using this model for area, you can experiment with different values of w to see how to obtain the maximum area. After trying several values, it appears that the maximum area occurs when

$$w = 6$$

as shown in the table.

Width, w	5.0	5.5	5.9	6.0	6.1	6.5	7.0
Area, A	35.00	35.75	35.99	36.00	35.99	35.75	35.00

In limit terminology, you can say that "the limit of A as w approaches 6 is 36." This is written as

$$\lim_{w \to 6} A = \lim_{w \to 6} (12w - w^2) = 36.$$

CHECKPOINT Now try Exercise 5.

Definition of Limit

Definition of Limit

If $f(x)$ becomes arbitrarily close to a unique number L as x approaches c from either side, then the **limit** of $f(x)$ as x approaches c is L. This is written as

$$\lim_{x \to c} f(x) = L.$$

Example 2 Estimating a Limit Numerically

Use a table to estimate the limit numerically.

$$\lim_{x \to 2} (3x - 2)$$

Solution

Let $f(x) = 3x - 2$. Then construct a table that shows values of $f(x)$ for two sets of x-values—one set that approaches 2 from the left and one that approaches 2 from the right.

x	1.9	1.99	1.999	2.0	2.001	2.01	2.1
$f(x)$	3.700	3.970	3.997	?	4.003	4.030	4.300

From the table, it appears that the closer x gets to 2, the closer $f(x)$ gets to 4. So, you can estimate the limit to be 4. Figure 11.2 adds further support to this conclusion.

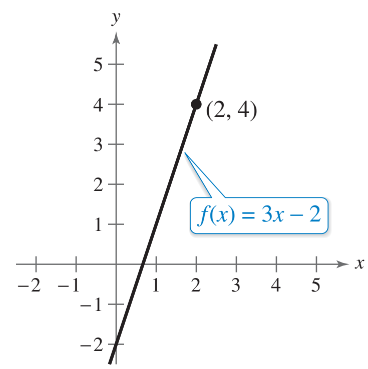


Figure 11.2

CHECKPOINT Now try Exercise 7.

In Figure 11.2, note that the graph of

$$f(x) = 3x - 2$$

is continuous. For graphs that are not continuous, finding a limit can be more difficult.

Example 3 Estimating a Limit Numerically

Use a table to estimate the limit numerically.

$$\lim_{x \to 0} \frac{x}{\sqrt{x+1} - 1}$$

Solution

Let $f(x) = x/\left(\sqrt{x+1} - 1\right)$. Then construct a table that shows values of $f(x)$ for two sets of x-values—one set that approaches 0 from the left and one that approaches 0 from the right.

x	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01
$f(x)$	1.99499	1.99949	1.99995	?	2.00005	2.00050	2.00499

From the table, it appears that the limit is 2. This limit is reinforced by the graph of f (see Figure 11.3).

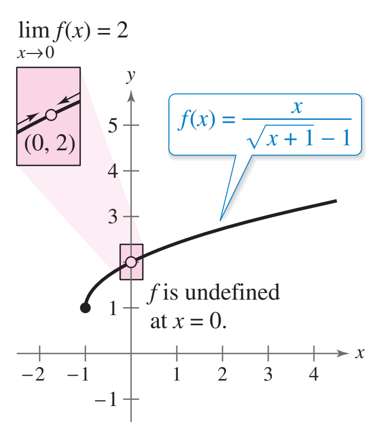


Figure 11.3

CHECKPOINT Now try Exercise 9.

In Example 3, note that $f(x)$ has a limit as $x \to 0$ even though the function is not defined at $x = 0$. This often happens, and it is important to realize that *the existence or nonexistence of $f(x)$ at $x = c$ has no bearing on the existence of the limit of $f(x)$ as x approaches c.*

Example 4 Using a Graphing Utility to Estimate a Limit

Estimate the limit.

$$\lim_{x \to 1} \frac{x^3 - x^2 + x - 1}{x - 1}$$

Numerical Solution

Let $f(x) = (x^3 - x^2 + x - 1)/(x - 1)$.

Create a table that shows values of $f(x)$ for several x-values near 1.

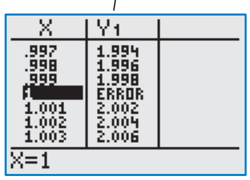


Figure 11.4

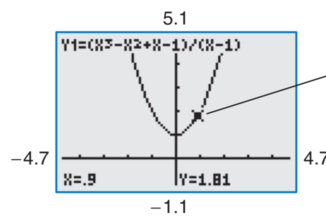
From Figure 11.4, it appears that the closer x gets to 1, the closer $f(x)$ gets to 2. So, you can estimate the limit to be 2.

Graphical Solution

Use a graphing utility to graph

$$f(x) = (x^3 - x^2 + x - 1)/(x - 1)$$

using a *decimal* setting, as shown in Figure 11.5.



Use the *trace* feature to determine that as x gets closer and closer to 1, $f(x)$ gets closer and closer to 2 from the left and from the right.

Figure 11.5

From Figure 11.5, you can estimate the limit to be 2. As you use the *trace* feature, notice that there is no value given for y when $x = 1$, and that there is a hole or break in the graph at $x = 1$.

CHECKPOINT Now try Exercise 15.

Example 5 Using a Graph to Find a Limit

Find the limit of $f(x)$ as x approaches 3, where f is defined as

$$f(x) = \begin{cases} 2, & x \neq 3 \\ 0, & x = 3 \end{cases}.$$

Some students may come to think that a limit is a quantity that can be approached but cannot actually be reached, as shown in Example 4. Remind them that some limits are like that, but, as Example 2 shows, many are not.

Solution

Because $f(x) = 2$ for all x other than $x = 3$ and because the value of $f(3)$ is immaterial, it follows that the limit is 2 (see Figure 11.6). So, you can write

$$\lim_{x \to 3} f(x) = 2.$$

The fact that $f(3) = 0$ has no bearing on the existence or value of the limit as x approaches 3. For instance, if the function were defined as

$$f(x) = \begin{cases} 2, & x \neq 3 \\ 4, & x = 3 \end{cases}$$

then the limit as x approaches 3 would be the same.

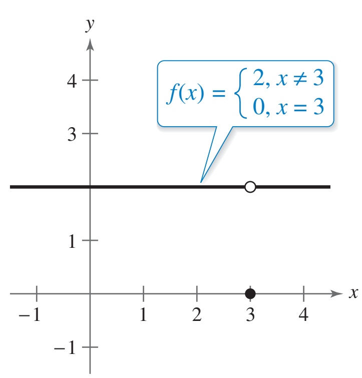


Figure 11.6

CHECKPOINT Now try Exercise 29.

Limits That Fail to Exist

In the next three examples, you will examine some limits that fail to exist.

Example 6 Comparing Left and Right Behavior

Show that the limit does not exist.

$$\lim_{x \to 0} \frac{|x|}{x}$$

Solution

Consider the graph of the function given by $f(x) = |x|/x$. In Figure 11.7, you can see that for positive x-values

$$\frac{|x|}{x} = 1, \quad x > 0$$

and for negative x-values

$$\frac{|x|}{x} = -1, \quad x < 0.$$

This means that no matter how close x gets to 0, there will be both positive and negative x-values that yield

$$f(x) = 1$$

and

$$f(x) = -1.$$

This implies that the limit does not exist.

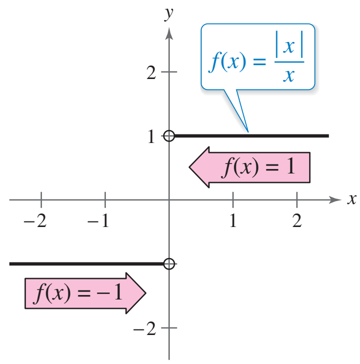


Figure 11.7

CHECKPOINT Now try Exercise 35.

What's Wrong?

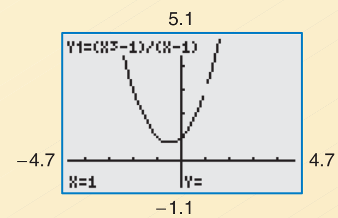
You use a graphing utility to graph

$$y_1 = \frac{x^3 - 1}{x - 1}$$

using a *decimal* setting, as shown in the figure. You use the *trace* feature to conclude that the limit

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$$

does not exist. What's wrong?



Consider reinforcing the nonexistence of the limits in Examples 6 and 7 by constructing and examining a table of values. Encourage students to investigate limits using a variety of approaches.

Example 7 Unbounded Behavior

Discuss the existence of the limit.

$$\lim_{x \to 0} \frac{1}{x^2}$$

Solution

Let $f(x) = 1/x^2$. In Figure 11.8, note that as x approaches 0 from either the right or the left, $f(x)$ increases without bound. This means that by choosing x close enough to 0, you can force $f(x)$ to be as large as you want. For instance, $f(x)$ will be larger than 100 when you choose x that is within $\frac{1}{10}$ of 0. That is,

$$0 < |x| < \frac{1}{10} \quad \Rightarrow \quad f(x) = \frac{1}{x^2} > 100.$$

Similarly, you can force $f(x)$ to be larger than 1,000,000, as follows.

$$0 < |x| < \frac{1}{1000} \quad \Rightarrow \quad f(x) = \frac{1}{x^2} > 1{,}000{,}000$$

Because $f(x)$ is not approaching a unique real number L as x approaches 0, you can conclude that the limit does not exist.

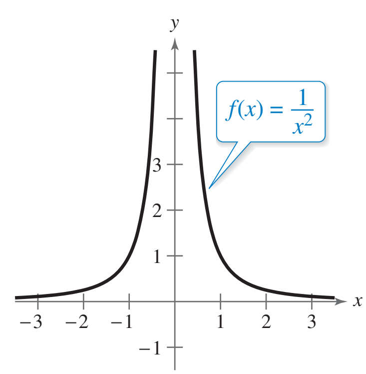


Figure 11.8

CHECKPOINT Now try Exercise 37.

Example 8 Oscillating Behavior

Discuss the existence of the limit.

$$\lim_{x \to 0} \sin \frac{1}{x}$$

Solution

Let $f(x) = \sin(1/x)$. In Figure 11.9, you can see that as x approaches 0, $f(x)$ oscillates between -1 and 1.

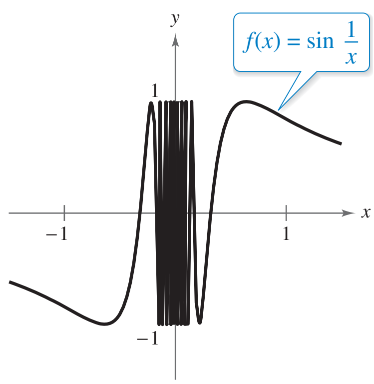


Figure 11.9

So, the limit does not exist because no matter how close you are to 0, it is possible to choose values of x_1 and x_2 such that

$$\sin \frac{1}{x_1} = 1 \quad \text{and} \quad \sin \frac{1}{x_2} = -1$$

as indicated in the table.

x	$\frac{2}{\pi}$	$\frac{2}{3\pi}$	$\frac{2}{5\pi}$	$\frac{2}{7\pi}$	$\frac{2}{9\pi}$	$\frac{2}{11\pi}$	$x \to 0$
$\sin \frac{1}{x}$	1	-1	1	-1	1	-1	Limit does not exist.

CHECKPOINT Now try Exercise 39.

Examples 6, 7, and 8 show three of the most common types of behavior associated with the *nonexistence* of a limit.

Conditions Under Which Limits Do Not Exist

The limit of $f(x)$ as $x \to c$ does not exist under any of the following conditions.

1. $f(x)$ approaches a different number from the right side of c than it approaches from the left side of c. Example 6
2. $f(x)$ increases or decreases without bound as x approaches c. Example 7
3. $f(x)$ oscillates between two fixed values as x approaches c. Example 8

Technology Tip

When using a graphing utility to investigate the behavior of a function near the x-value at which you are trying to evaluate a limit, remember that you cannot always trust the graphs that the graphing utility displays. For instance, consider the incorrect graph shown in Figure 11.10. The graphing utility can't show the correct graph because $f(x) = \sin(1/x)$ has infinitely many oscillations over any interval that contains 0.

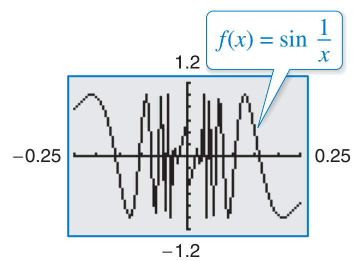


Figure 11.10

Properties of Limits and Direct Substitution

You have seen that sometimes the limit of $f(x)$ as $x \to c$ is simply $f(c)$. In such cases, it is said that the limit can be evaluated by **direct substitution.** That is,

$$\lim_{x \to c} f(x) = f(c). \qquad \text{Substitute } c \text{ for } x.$$

There are many "well-behaved" functions, such as polynomial functions and rational functions with nonzero denominators, that have this property. Some of the basic ones are included in the following list.

Basic Limits

Let b and c be real numbers and let n be a positive integer.

1. $\lim\limits_{x \to c} b = b$
2. $\lim\limits_{x \to c} x = c$
3. $\lim\limits_{x \to c} x^n = c^n$ (See the proof on page 804.)
4. $\lim\limits_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$, for n even and $c > 0$

Trigonometric functions can also be included in this list. For instance,

$$\lim_{x \to \pi} \sin x = \sin \pi = 0 \quad \text{and} \quad \lim_{x \to 0} \cos x = \cos 0 = 1.$$

By combining the basic limits with the following operations, you can find limits for a wide variety of functions.

Properties of Limits

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = K$$

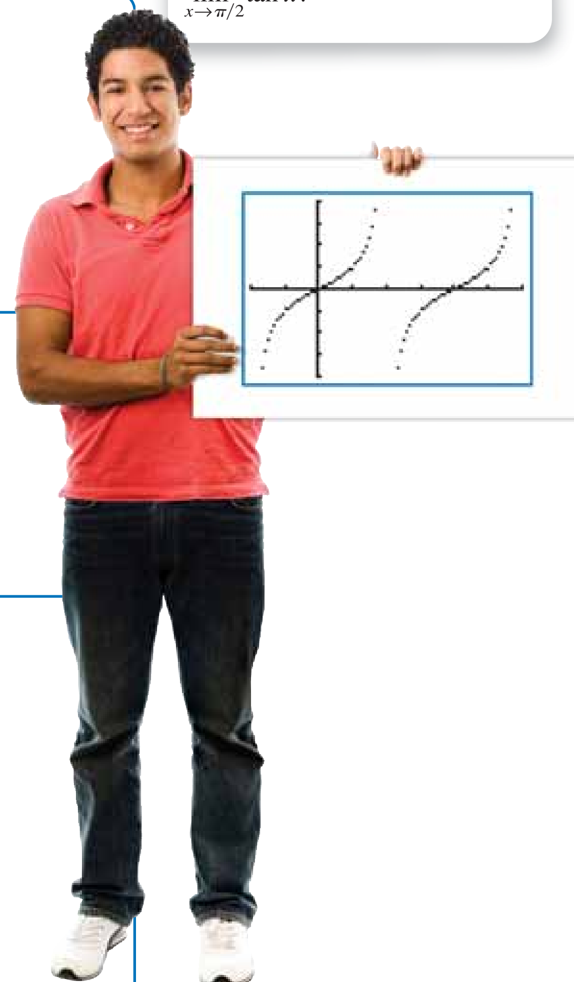
1. Scalar multiple: $\lim\limits_{x \to c} [bf(x)] = bL$
2. Sum or difference: $\lim\limits_{x \to c} [f(x) \pm g(x)] = L \pm K$
3. Product: $\lim\limits_{x \to c} [f(x)g(x)] = LK$
4. Quotient: $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}$, provided $K \neq 0$
5. Power: $\lim\limits_{x \to c} [f(x)]^n = L^n$

Technology Tip

When evaluating limits, remember that there are several ways to solve most problems. Often, a problem can be solved *numerically*, *graphically*, or *algebraically*. You can use a graphing utility to confirm the limits in the examples and in the exercise set numerically using the *table* feature or graphically using the *zoom* and *trace* features.

Explore the Concept

Use a graphing utility to graph the tangent function. What are $\lim\limits_{x \to 0} \tan x$ and $\lim\limits_{x \to \pi/4} \tan x$? What can you say about the existence of the limit $\lim\limits_{x \to \pi/2} \tan x$?



Additional Example

Let $\lim\limits_{x \to 3} f(x) = 7$ and $\lim\limits_{x \to 3} g(x) = 12$.

a. $\lim\limits_{x \to 3} [f(x) - g(x)] = -5$

b. $\lim\limits_{x \to 3} [f(x)g(x)] = 84$

c. $\lim\limits_{x \to 3} [g(x)]^{1/2} = 2\sqrt{3}$

Example 9 Direct Substitution and Properties of Limits

a. $\lim\limits_{x\to 4} x^2 = (4)^2 = 16$ — Direct Substitution

b. $\lim\limits_{x\to 4} 5x = 5\lim\limits_{x\to 4} x = 5(4) = 20$ — Scalar Multiple Property

c. $\lim\limits_{x\to \pi} \dfrac{\tan x}{x} = \dfrac{\lim\limits_{x\to\pi} \tan x}{\lim\limits_{x\to\pi} x} = \dfrac{0}{\pi} = 0$ — Quotient Property

d. $\lim\limits_{x\to 9} \sqrt{x} = \sqrt{9} = 3$ — Direct Substitution

e. $\lim\limits_{x\to \pi} (x\cos x) = \left(\lim\limits_{x\to\pi} x\right)\left(\lim\limits_{x\to\pi} \cos x\right)$ — Product Property

$$= \pi(\cos \pi)$$

$$= -\pi$$

f. $\lim\limits_{x\to 3} (x+4)^2 = \left[\left(\lim\limits_{x\to 3} x\right) + \left(\lim\limits_{x\to 3} 4\right)\right]^2$ — Sum and Power Properties

$$= (3+4)^2$$

$$= 7^2 = 49$$

CHECKPOINT Now try Exercise 51.

Explore the Concept

Sketch the graph of each function. Then find the limits of each function as x approaches 1 and as x approaches 2. What conclusions can you make?

a. $f(x) = x + 1$

b. $g(x) = \dfrac{x^2 - 1}{x - 1}$

c. $h(x) = \dfrac{x^3 - 2x^2 - x + 2}{x^2 - 3x + 2}$

Use a graphing utility to graph each function above. Does the graphing utility distinguish among the three graphs? Write a short explanation of your findings.

The results of using direct substitution to evaluate limits of polynomial and rational functions are summarized as follows.

Limits of Polynomial and Rational Functions

1. If p is a polynomial function and c is a real number, then

$$\lim_{x\to c} p(x) = p(c).$$ (See the proof on page 804.)

2. If r is a rational function given by $r(x) = p(x)/q(x)$, and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x\to c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

Example 10 Evaluating Limits by Direct Substitution

Find each limit.

a. $\lim\limits_{x\to -1} (x^2 + x - 6)$ **b.** $\lim\limits_{x\to -1} \dfrac{x^2 + x - 6}{x + 3}$

Solution

a. To evaluate the limit of a polynomial function, use direct substitution.

$$\lim_{x\to -1} (x^2 + x - 6) = (-1)^2 + (-1) - 6 = -6$$

b. The denominator is not 0 when $x = -1$, so you can evaluate the limit of the rational function using direct substitution.

$$\lim_{x\to -1} \frac{x^2 + x - 6}{x + 3} = \frac{(-1)^2 + (-1) - 6}{-1 + 3} = -\frac{6}{2} = -3$$

CHECKPOINT Now try Exercise 55.

Explore the Concept

Use a graphing utility to graph the function

$$f(x) = \frac{x^2 - 3x - 10}{x - 5}.$$

Use the *trace* feature to approximate $\lim\limits_{x\to 4} f(x)$. What do you think $\lim\limits_{x\to 5} f(x)$ equals? Is f defined at $x = 5$? Does this affect the existence of the limit as x approaches 5?

11.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blank.

1. If $f(x)$ becomes arbitrarily close to a unique number L as x approaches c from either side, then the ________ of $f(x)$ as x approaches c is L.

2. To find a limit of a polynomial function, use ________ .

3. Find the limit: $\lim\limits_{x \to 0} 3$.

4. List three conditions under which limits do not exist.

Procedures and Problem Solving

5. *Why you should learn it* *(p. 750)* You create an open box from a square piece of material, 24 centimeters on a side. You cut equal squares from the corners and turn up the sides.

(a) Draw and label a diagram that represents the box.

(b) Verify that the volume V of the box is given by $V = 4x(12 - x)^2$.

(c) The box has a maximum volume when $x = 4$. Use a graphing utility to complete the table and observe the behavior of the function as x approaches 4. Use the table to find $\lim\limits_{x \to 4} V$.

x	3	3.5	3.9	4	4.1	4.5	5
V							

(d) Use the graphing utility to graph the volume function. Verify that the volume is maximum when $x = 4$.

6. **Landscape Design** A landscaper arranges bricks to enclose a region shaped like a right triangle with a hypotenuse of $\sqrt{18}$ meters whose area is as large as possible.

(a) Draw and label a diagram that shows the base x and height y of the triangle.

(b) Verify that the area A of the triangle is given by

$$A = \tfrac{1}{2}x\sqrt{18 - x^2}.$$

(c) The triangle has a maximum area when $x = 3$ meters. Use a graphing utility to complete the table and observe the behavior of the function as x approaches 3. Use the table to find $\lim\limits_{x \to 3} A$.

x	2	2.5	2.9	3	3.1	3.5	4
A							

(d) Use the graphing utility to graph the area function. Verify that the area is maximum when $x = 3$ meters.

Estimating a Limit Numerically **In Exercises 7–12, complete the table and use the result to estimate the limit numerically. Determine whether the limit can be reached.**

7. $\lim\limits_{x \to 2} (5x + 4)$

x	1.9	1.99	1.999	2	2.001	2.01	2.1
$f(x)$				?			

8. $\lim\limits_{x \to 1} (2x^2 + x - 4)$

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$				?			

9. $\lim\limits_{x \to -1} \dfrac{x + 1}{x^2 - x - 2}$

x	-1.1	-1.01	-1.001	-1	-0.999
$f(x)$				?	

x	-0.99	-0.9
$f(x)$		

10. $\lim\limits_{x \to 0} \dfrac{\sin 2x}{x}$

x	-0.1	-0.01	-0.001	0	0.001
$f(x)$				?	

x	0.01	0.1
$f(x)$		

11. $\lim_{x \to 0} \frac{\tan x}{2x}$

x	-0.1	-0.01	-0.001	0	0.001
$f(x)$				?	

x	0.01	0.1
$f(x)$		

12. $\lim_{x \to 1} \frac{\ln x}{x - 1}$

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$				?			

Using a Graphing Utility to Estimate a Limit **In Exercises 13–28, use the *table* feature of a graphing utility to create a table for the function and use the result to estimate the limit numerically. Use the graphing utility to graph the corresponding function to confirm your result graphically.**

13. $\lim_{x \to -1} \frac{x^2 - 1}{x + 1}$

14. $\lim_{x \to 2} \frac{4 - x^2}{x - 2}$

15. $\lim_{x \to 1} \frac{x - 1}{x^2 + 2x - 3}$

16. $\lim_{x \to -2} \frac{x + 2}{x^2 + 5x + 6}$

17. $\lim_{x \to 0} \frac{\sqrt{x + 5} - \sqrt{5}}{x}$

18. $\lim_{x \to -3} \frac{\sqrt{1 - x} - 2}{x + 3}$

19. $\lim_{x \to -4} \frac{\frac{x}{x + 2} - 2}{x + 4}$

20. $\lim_{x \to 2} \frac{\frac{1}{x + 2} - \frac{1}{4}}{x - 2}$

21. $\lim_{x \to 0} \frac{\sin x}{x}$

22. $\lim_{x \to 0} \frac{\cos x - 1}{x}$

23. $\lim_{x \to 0} \frac{\sin^2 x}{x}$

24. $\lim_{x \to 0} \frac{2x}{\tan 4x}$

25. $\lim_{x \to 0} \frac{e^{2x} - 1}{2x}$

26. $\lim_{x \to 0} \frac{1 - e^{-4x}}{x}$

27. $\lim_{x \to 2} \frac{\ln(2x - 3)}{x - 2}$

28. $\lim_{x \to 1} \frac{\ln(x^2)}{x - 1}$

Using a Graph to Find a Limit **In Exercises 29–32, graph the function and find the limit (if it exists) as x approaches 2.**

29. $f(x) = \begin{cases} 3, & x \neq 2 \\ 1, & x = 2 \end{cases}$

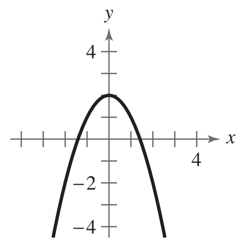
30. $f(x) = \begin{cases} x, & x \neq 2 \\ -4, & x = 2 \end{cases}$

31. $f(x) = \begin{cases} 2x + 1, & x < 2 \\ x + 3, & x \geq 2 \end{cases}$

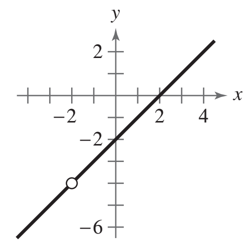
32. $f(x) = \begin{cases} -2x, & x \leq 2 \\ x^2 - 4x + 1, & x > 2 \end{cases}$

Using a Graph to Find a Limit **In Exercises 33–40, use the graph to find the limit (if it exists). If the limit does not exist, explain why.**

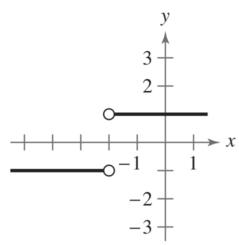
33. $\lim_{x \to 0} (2 - x^2)$



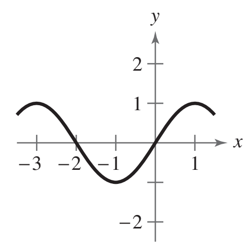
34. $\lim_{x \to -2} \frac{x^2 - 4}{x + 2}$



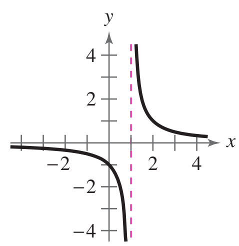
35. $\lim_{x \to -2} \frac{|x + 2|}{x + 2}$



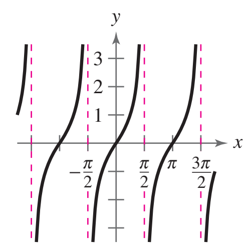
36. $\lim_{x \to -1} \sin \frac{\pi x}{2}$



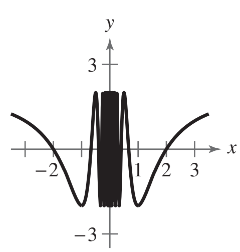
37. $\lim_{x \to 1} \frac{1}{x - 1}$



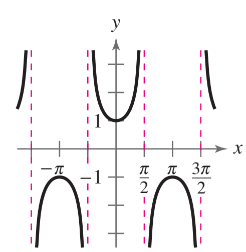
38. $\lim_{x \to \pi/2} \tan x$



39. $\lim_{x \to 0} 2 \cos \frac{\pi}{x}$



40. $\lim_{x \to \pi/2} \sec x$



Determining Whether a Limit Exists **In Exercises 41–48, use a graphing utility to graph the function and use the graph to determine whether the limit exists. If the limit does not exist, explain why.**

41. $f(x) = \frac{5}{2 + e^{1/x}}, \quad \lim_{x \to 0} f(x)$

42. $f(x) = \frac{e^x - 1}{x}, \quad \lim_{x \to 0} f(x)$

43. $f(x) = \cos \frac{1}{x}, \quad \lim_{x\to 0} f(x)$

44. $f(x) = \sin \pi x, \quad \lim_{x\to -1} f(x)$

45. $f(x) = \dfrac{\sqrt{x+3} - 1}{x - 4}, \quad \lim_{x\to 4} f(x)$

46. $f(x) = \dfrac{\sqrt{x+5} - 4}{x - 2}, \quad \lim_{x\to 2} f(x)$

47. $f(x) = \ln(x + 3), \quad \lim_{x\to 4} f(x)$

48. $f(x) = \ln(7 - x), \quad \lim_{x\to -1} f(x)$

Evaluating Limits **In Exercises 49 and 50, use the given information to evaluate each limit.**

49. $\lim_{x\to c} f(x) = 4, \quad \lim_{x\to c} g(x) = 8$

(a) $\lim_{x\to c} [-2g(x)]$ (b) $\lim_{x\to c} [f(x) + g(x)]$

(c) $\lim_{x\to c} \dfrac{f(x)}{g(x)}$ (d) $\lim_{x\to c} \sqrt{f(x)}$

50. $\lim_{x\to c} f(x) = 3, \quad \lim_{x\to c} g(x) = -1$

(a) $\lim_{x\to c} [f(x) + g(x)]^2$ (b) $\lim_{x\to c} [6f(x)g(x)]$

(c) $\lim_{x\to c} \dfrac{5g(x)}{4f(x)}$ (d) $\lim_{x\to c} \dfrac{1}{\sqrt{f(x)}}$

Evaluating Limits **In Exercises 51 and 52, find (a) $\lim_{x\to 2} f(x)$, (b) $\lim_{x\to 2} g(x)$, (c) $\lim_{x\to 2} [f(x)g(x)]$, and (d) $\lim_{x\to 2} [g(x) - f(x)]$.**

51. $f(x) = x^3, \quad g(x) = \dfrac{\sqrt{x^2+5}}{2x^2}$

52. $f(x) = \dfrac{x}{3 - x}, \quad g(x) = \sin \pi x$

Evaluating a Limit by Direct Substitution **In Exercises 53–68, find the limit by direct substitution.**

53. $\lim_{x\to 5} (10 - x^2)$

54. $\lim_{x\to -2} \left(\frac{1}{2}x^3 - 5x\right)$

55. $\lim_{x\to -3} (2x^2 + 4x + 1)$

56. $\lim_{x\to -2} (x^3 - 6x + 5)$

57. $\lim_{x\to -3} \dfrac{3x}{x^2 + 1}$

58. $\lim_{x\to 4} \dfrac{x - 1}{x^2 + 2x + 3}$

59. $\lim_{x\to -2} \dfrac{5x + 3}{2x - 9}$

60. $\lim_{x\to 3} \dfrac{x^2 + 1}{x}$

61. $\lim_{x\to -1} \sqrt{x + 2}$

62. $\lim_{x\to 3} \sqrt[3]{x^2 - 1}$

63. $\lim_{x\to 3} e^x$

64. $\lim_{x\to e} \ln x$

65. $\lim_{x\to \pi} \sin 2x$

66. $\lim_{x\to \pi} \tan x$

67. $\lim_{x\to 1/2} \arcsin x$

68. $\lim_{x\to 1} \arccos \dfrac{x}{2}$

Conclusions

True or False? **In Exercises 69 and 70, determine whether the statement is true or false. Justify your answer.**

69. The limit of a function as x approaches c does not exist when the function approaches -3 from the left of c and 3 from the right of c.

70. If f is a rational function, then the limit of $f(x)$ as x approaches c is $f(c)$.

71. **Think About It** From Exercises 7 to 12, select a limit that can be reached and one that cannot be reached.

(a) Use a graphing utility to graph the corresponding functions using a standard viewing window. Do the graphs reveal whether the limit can be reached? Explain.

(b) Use the graphing utility to graph the corresponding functions using a *decimal* setting. Do the graphs reveal whether the limit can be reached? Explain.

72. **Think About It** Use the results of Exercise 71 to draw a conclusion as to whether you can use the graph generated by a graphing utility to determine reliably when a limit can be reached.

73. **Think About It**

(a) Given $f(2) = 4$, can you conclude anything about $\lim_{x\to 2} f(x)$? Explain your reasoning.

(b) Given $\lim_{x\to 2} f(x) = 4$, can you conclude anything about $f(2)$? Explain your reasoning.

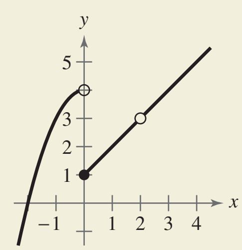
74. **CAPSTONE** Use the graph of the function f to decide whether the value of the given quantity exists. If it does, find it. If not, explain why.

(a) $f(0)$

(b) $\lim_{x\to 0} f(x)$

(c) $f(2)$

(d) $\lim_{x\to 2} f(x)$



Cumulative Mixed Review

Simplifying Rational Expressions **In Exercises 75–80, simplify the rational expression.**

75. $\dfrac{5 - x}{3x - 15}$

76. $\dfrac{x^2 - 81}{9 - x}$

77. $\dfrac{15x^2 + 7x - 4}{15x^2 + x - 2}$

78. $\dfrac{x^2 - 12x + 36}{x^2 - 7x + 6}$

79. $\dfrac{x^3 + 27}{x^2 + x - 6}$

80. $\dfrac{x^3 - 8}{x^2 - 4}$

11.2 Techniques for Evaluating Limits

Dividing Out Technique

In Section 11.1, you studied several types of functions whose limits can be evaluated by direct substitution. In this section, you will study several techniques for evaluating limits of functions for which direct substitution fails.

Suppose you were asked to find the following limit.

$$\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3}$$

Direct substitution fails because -3 is a zero of the denominator. By using a table, however, it appears that the limit of the function as x approaches -3 is -5.

x	-3.01	-3.001	-3.0001	-3	-2.9999	-2.999	-2.99
$\frac{x^2 + x - 6}{x + 3}$	-5.01	-5.001	-5.0001	?	-4.9999	-4.999	-4.99

Another way to find the limit of this function is shown in Example 1.

Example 1 Dividing Out Technique

Find the limit.

$$\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3}$$

Solution

Begin by factoring the numerator and dividing out any common factors.

$$\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3} = \lim_{x \to -3} \frac{(x - 2)(x + 3)}{x + 3} \qquad \text{Factor numerator.}$$

$$= \lim_{x \to -3} \frac{(x - 2)(\cancel{x + 3})}{\cancel{x + 3}} \qquad \text{Divide out common factor.}$$

$$= \lim_{x \to -3} (x - 2) \qquad \text{Simplify.}$$

$$= -3 - 2 \qquad \text{Direct substitution}$$

$$= -5 \qquad \text{Simplify.}$$

CHECKPOINT Now try Exercise 11.

This procedure for evaluating a limit is called the **dividing out technique.** The validity of this technique stems from the fact that when two functions agree at all but a single number c, they must have identical limit behavior at $x = c$. In Example 1, the functions given by

$$f(x) = \frac{x^2 + x - 6}{x + 3} \qquad \text{and} \qquad g(x) = x - 2$$

agree at all values of x other than

$$x = -3.$$

So, you can use $g(x)$ to find the limit of $f(x)$.

What you should learn

- Use the dividing out technique to evaluate limits of functions.
- Use the rationalizing technique to evaluate limits of functions.
- Use technology to approximate limits of functions graphically and numerically.
- Evaluate one-sided limits of functions.
- Evaluate limits of difference quotients from calculus.

Why you should learn it

Many definitions in calculus involve the limit of a function. For instance, in Exercises 69 and 70 on page 768, the definition of the velocity of a free-falling object at any instant in time involves finding the limit of a position function.

The dividing out technique should be applied only when direct substitution produces 0 in both the numerator *and* the denominator. An expression such as $\frac{0}{0}$ has no meaning as a real number. It is called an **indeterminate form** because you cannot, from the form alone, determine the limit. When you try to evaluate a limit of a rational function by direct substitution and encounter this form, you can conclude that the numerator and denominator must have a common factor. After factoring and dividing out, you should try direct substitution again.

Example 2 Dividing Out Technique

Find the limit.

$$\lim_{x \to 1} \frac{x - 1}{x^3 - x^2 + x - 1}$$

Consider suggesting to your students that they try making a table of values to estimate the limit in Example 2 before finding it algebraically. A range of 0.9 through 1.1 with increment 0.01 is useful.

Solution

Begin by substituting $x = 1$ into the numerator and denominator.

$$\frac{x - 1}{x^3 - x^2 + x - 1} \qquad \begin{array}{ll} 1 - 1 = 0 & \text{Numerator is 0 when } x = 1. \\ 1^3 - 1^2 + 1 - 1 = 0 & \text{Denominator is 0 when } x = 1. \end{array}$$

Because both the numerator and denominator are zero when $x = 1$, direct substitution will not yield the limit. To find the limit, you should factor the numerator and denominator, divide out any common factors, and then try direct substitution again.

$$\begin{aligned}
\lim_{x \to 1} \frac{x - 1}{x^3 - x^2 + x - 1} &= \lim_{x \to 1} \frac{x - 1}{(x - 1)(x^2 + 1)} && \text{Factor denominator.} \\
&= \lim_{x \to 1} \frac{\cancel{x - 1}}{\cancel{(x - 1)}(x^2 + 1)} && \text{Divide out common factor.} \\
&= \lim_{x \to 1} \frac{1}{x^2 + 1} && \text{Simplify.} \\
&= \frac{1}{1^2 + 1} && \text{Direct substitution} \\
&= \frac{1}{2} && \text{Simplify.}
\end{aligned}$$

This result is shown graphically in Figure 11.11.

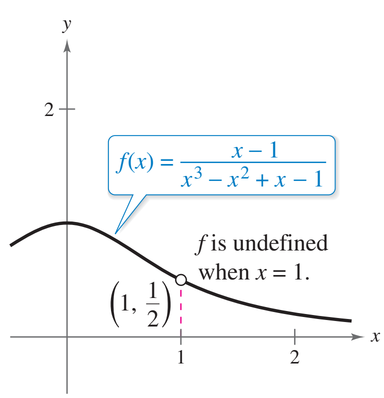


Figure 11.11

CHECKPOINT Now try Exercise 15.

Study Tip

In Example 2, the factorization of the denominator can be obtained by dividing by $(x - 1)$ or by grouping as follows.

$$\begin{aligned}
&x^3 - x^2 + x - 1 \\
&\quad = x^2(x - 1) + (x - 1) \\
&\quad = (x - 1)(x^2 + 1)
\end{aligned}$$

Rationalizing Technique

Another way to find the limits of some functions is first to rationalize the numerator of the function. This is called the **rationalizing technique.** Recall that rationalizing the numerator means multiplying the numerator and denominator by the conjugate of the numerator. For instance, the conjugate of $\sqrt{x} + 4$ is

$$\sqrt{x} - 4.$$

Example 3 Rationalizing Technique

Find the limit.

$$\lim_{x\to 0} \frac{\sqrt{x+1} - 1}{x}$$

Solution

By direct substitution, you obtain the indeterminate form $\frac{0}{0}$.

$$\lim_{x\to 0} \frac{\sqrt{x+1} - 1}{x} = \frac{\sqrt{0+1} - 1}{0} = \frac{0}{0} \qquad \text{Indeterminate form}$$

In this case, you can rewrite the fraction by rationalizing the numerator.

$$\begin{aligned}
\frac{\sqrt{x+1} - 1}{x} &= \left(\frac{\sqrt{x+1} - 1}{x}\right)\left(\frac{\sqrt{x+1} + 1}{\sqrt{x+1} + 1}\right) \\
&= \frac{(x+1) - 1}{x\left(\sqrt{x+1} + 1\right)} && \text{Multiply.} \\
&= \frac{x}{x\left(\sqrt{x+1} + 1\right)} && \text{Simplify.} \\
&= \frac{\cancel{x}}{\cancel{x}\left(\sqrt{x+1} + 1\right)} && \text{Divide out common factor.} \\
&= \frac{1}{\sqrt{x+1} + 1}, \quad x \neq 0 && \text{Simplify.}
\end{aligned}$$

Now you can evaluate the limit by direct substitution.

$$\lim_{x\to 0} \frac{\sqrt{x+1} - 1}{x} = \lim_{x\to 0} \frac{1}{\sqrt{x+1} + 1} = \frac{1}{\sqrt{0+1} + 1} = \frac{1}{1+1} = \frac{1}{2}$$

You can reinforce your conclusion that the limit is $\frac{1}{2}$ by constructing a table, as shown below, or by sketching a graph, as shown in Figure 11.12.

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0.5132	0.5013	0.5001	?	0.4999	0.4988	0.4881

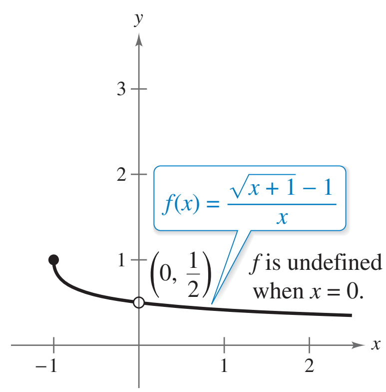


Figure 11.12

CHECKPOINT Now try Exercise 23.

The rationalizing technique for evaluating limits is based on multiplication by a convenient form of 1. In Example 3, the convenient form is

$$1 = \frac{\sqrt{x+1} + 1}{\sqrt{x+1} + 1}.$$

Using Technology

The dividing out and rationalizing techniques may not work well for finding limits of nonalgebraic functions. You often need to use more sophisticated analytic techniques to find limits of these types of functions.

Example 4 Approximating a Limit Numerically

Approximate the limit: $\lim_{x\to 0} (1 + x)^{1/x}$.

Solution

Let $f(x) = (1 + x)^{1/x}$.

Create a table that shows values of $f(x)$ for several x-values near 0.

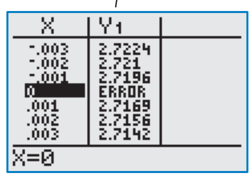


Figure 11.13

Because 0 is halfway between -0.001 and 0.001 (see Figure 11.13), use the average of the values of f at these two x-coordinates to estimate the limit.

$$\lim_{x\to 0} (1 + x)^{1/x} \approx \frac{2.7196 + 2.7169}{2} = 2.71825$$

The actual limit can be found algebraically to be

$$e \approx 2.71828.$$

CHECKPOINT Now try Exercise 37.

Example 5 Approximating a Limit Graphically

Approximate the limit: $\lim_{x\to 0} \frac{\sin x}{x}$.

Solution

Direct substitution produces the indeterminate form $\frac{0}{0}$. To approximate the limit, begin by using a graphing utility to graph $f(x) = (\sin x)/x$, as shown in Figure 11.14. Then use the *zoom* and *trace* features of the graphing utility to choose a point on each side of 0, such as $(-0.0012467, 0.9999997)$ and $(0.0012467, 0.9999997)$. Because 0 is halfway between -0.0012467 and 0.0012467, use the average of the values of f at these two x-coordinates to estimate the limit.

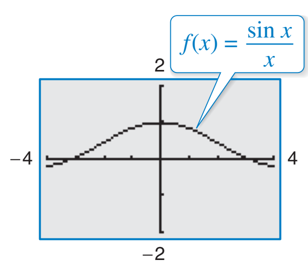


Figure 11.14

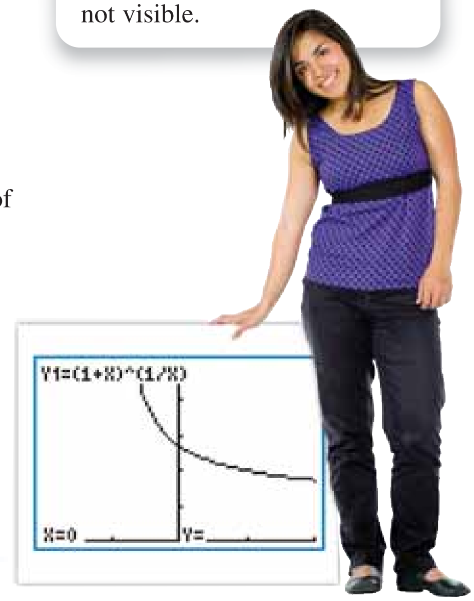
$$\lim_{x\to 0} \frac{\sin x}{x} \approx \frac{0.9999997 + 0.9999997}{2} = 0.9999997$$

It can be shown algebraically that this limit is exactly 1.

CHECKPOINT Now try Exercise 43.

Technology Tip

In Example 4, a graph of $f(x) = (1 + x)^{1/x}$ on a graphing utility would appear continuous at $x = 0$ (see below). But when you try to use the *trace* feature of a graphing utility to determine $f(0)$, no value is given. Some graphing utilities can show breaks or holes in a graph when an appropriate viewing window is used. Because the hole in the graph of f occurs on the y-axis, the hole is not visible.



One-Sided Limits

In Section 11.1, you saw that one way in which a limit can fail to exist is when a function approaches a different value from the left side of c than it approaches from the right side of c. This type of behavior can be described more concisely with the concept of a **one-sided limit.**

$$\lim_{x \to c^-} f(x) = L_1 \text{ or } f(x) \to L_1 \text{ as } x \to c^- \qquad \text{Limit from the left}$$

$$\lim_{x \to c^+} f(x) = L_2 \text{ or } f(x) \to L_2 \text{ as } x \to c^+ \qquad \text{Limit from the right}$$

Example 6 Evaluating One-Sided Limits

You might wish to illustrate the concept of one-sided limits (and why they are necessary) with tables or graphs.

Find the limit as $x \to 0$ from the left and the limit as $x \to 0$ from the right for

$$f(x) = \frac{|2x|}{x}.$$

Solution

From the graph of f, shown in Figure 11.15, you can see that $f(x) = -2$ for all $x < 0$.

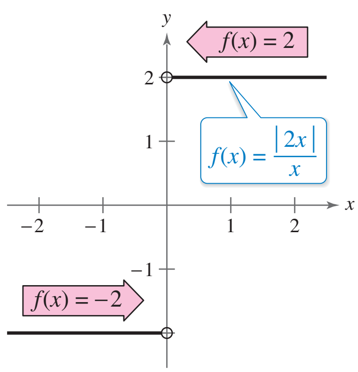


Figure 11.15

So, the limit from the left is

$$\lim_{x \to 0^-} \frac{|2x|}{x} = -2. \qquad \text{Limit from the left}$$

Because $f(x) = 2$ for all $x > 0$, the limit from the right is

$$\lim_{x \to 0^+} \frac{|2x|}{x} = 2. \qquad \text{Limit from the right}$$

CHECKPOINT Now try Exercise 49.

In Example 6, note that the function approaches different limits from the left and from the right. In such cases, the limit of $f(x)$ as $x \to c$ does not exist. For the limit of a function to exist as $x \to c$, it must be true that both one-sided limits exist and are equal.

Existence of a Limit

If f is a function and c and L are real numbers, then

$$\lim_{x \to c} f(x) = L$$

if and only if both the left and right limits *exist* and are *equal* to L.

Example 7 Finding One-Sided Limits

Find the limit of $f(x)$ as x approaches 1.

$$f(x) = \begin{cases} 4 - x, & x < 1 \\ 4x - x^2, & x > 1 \end{cases}$$

Solution

Remember that you are concerned about the value of f *near* $x = 1$ rather than *at* $x = 1$. So, for $x < 1$, $f(x)$ is given by $4 - x$, and you can use direct substitution to obtain

$$\begin{aligned} \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} (4 - x) \\ &= 4 - 1 \\ &= 3. \end{aligned}$$

For $x > 1$, $f(x)$ is given by $4x - x^2$, and you can use direct substitution to obtain

$$\begin{aligned} \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} (4x - x^2) \\ &= 4(1) - 1^2 \\ &= 3. \end{aligned}$$

Because the one-sided limits both exist and are equal to 3, it follows that

$$\lim_{x \to 1} f(x) = 3.$$

The graph in Figure 11.16 confirms this conclusion.

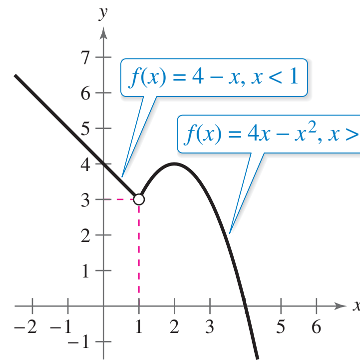


Figure 11.16

CHECKPOINT Now try Exercise 53.

Example 8 Comparing Limits from the Left and Right

To ship a package overnight, a delivery service charges \$17.80 for the first pound and \$1.40 for each additional pound or portion of a pound. Let x represent the weight of a package and let $f(x)$ represent the shipping cost. Show that the limit of $f(x)$ as $x \to 2$ does not exist.

$$f(x) = \begin{cases} 17.80, & 0 < x \le 1 \\ 19.20, & 1 < x \le 2 \\ 20.60, & 2 < x \le 3 \end{cases}$$

Solution

The graph of f is shown in Figure 11.17. The limit of $f(x)$ as x approaches 2 from the left is

$$\lim_{x \to 2^-} f(x) = 19.20$$

whereas the limit of $f(x)$ as x approaches 2 from the right is

$$\lim_{x \to 2^+} f(x) = 20.60.$$

Because these one-sided limits are not equal, the limit of $f(x)$ as $x \to 2$ does not exist.

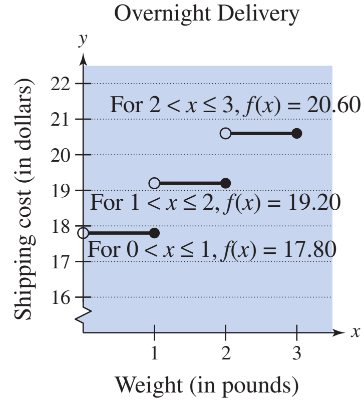


Figure 11.17

CHECKPOINT Now try Exercise 71.

A Limit from Calculus

In the next section, you will study an important type of limit from calculus—the limit of a *difference quotient*.

Example 9 Evaluating a Limit from Calculus

algebra of calculus

For the function given by $f(x) = x^2 - 1$, find

$$\lim_{h\to 0} \frac{f(3+h) - f(3)}{h}.$$

Solution

Direct substitution produces an indeterminate form.

$$\begin{aligned}
\lim_{h\to 0} \frac{f(3+h) - f(3)}{h} &= \lim_{h\to 0} \frac{[(3+h)^2 - 1] - [(3)^2 - 1]}{h} \\
&= \lim_{h\to 0} \frac{9 + 6h + h^2 - 1 - 9 + 1}{h} \\
&= \lim_{h\to 0} \frac{6h + h^2}{h} \\
&= \frac{0}{0}
\end{aligned}$$

By factoring and dividing out, you obtain the following.

$$\begin{aligned}
\lim_{h\to 0} \frac{f(3+h) - f(3)}{h} &= \lim_{h\to 0} \frac{6h + h^2}{h} \\
&= \lim_{h\to 0} \frac{h(6+h)}{h} \\
&= \lim_{h\to 0} (6+h) \\
&= 6 + 0 \\
&= 6
\end{aligned}$$

So, the limit is 6.

CHECKPOINT Now try Exercise 79.

Example 9 previews the derivative that is introduced in Section 11.3.

Group Activity
Write a limit problem (be sure the limit exists) and exchange it with that of a partner. Use a numerical approach to estimate the limit, and use an algebraic approach to verify your estimate. Discuss your results with your partner.

Note that for any x-value, the limit of a difference quotient is an expression of the form

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Direct substitution into the difference quotient always produces the indeterminate form $\frac{0}{0}$. For instance,

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \frac{f(x+0) - f(x)}{0} \\
&= \frac{f(x) - f(x)}{0} \\
&= \frac{0}{0}.
\end{aligned}$$

11.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blank.

1. To find a limit of a rational function that has common factors in its numerator and denominator, use the ________ .

2. The expression $\frac{0}{0}$ has no meaning as a real number and is called an ________ because you cannot, from the form alone, determine the limit.

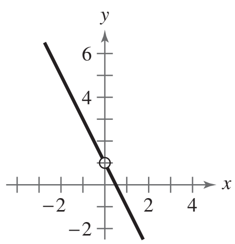
3. Which algebraic technique can you use to find $\lim_{x \to 0} \frac{\sqrt{x+4}-2}{x}$?

4. Describe in words what is meant by $\lim_{x \to 0^+} f(x) = -2$.

Procedures and Problem Solving

Using a Graph to Determine Limits **In Exercises 5–8, use the graph to determine each limit (if it exists). Then identify another function that agrees with the given function at all but one point.**

5. $g(x) = \dfrac{-2x^2 + x}{x}$

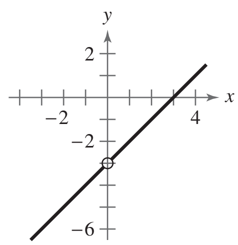


(a) $\lim_{x \to 0} g(x)$

(b) $\lim_{x \to -1} g(x)$

(c) $\lim_{x \to -2} g(x)$

6. $h(x) = \dfrac{x^2 - 3x}{x}$

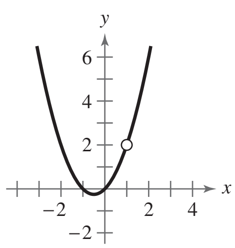


(a) $\lim_{x \to -2} h(x)$

(b) $\lim_{x \to 0} h(x)$

(c) $\lim_{x \to 3} h(x)$

7. $g(x) = \dfrac{x^3 - x}{x - 1}$

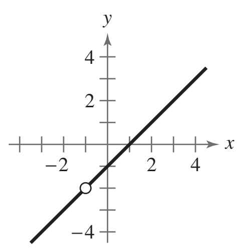


(a) $\lim_{x \to 1} g(x)$

(b) $\lim_{x \to -1} g(x)$

(c) $\lim_{x \to 0} g(x)$

8. $f(x) = \dfrac{x^2 - 1}{x + 1}$



(a) $\lim_{x \to 1} f(x)$

(b) $\lim_{x \to 2} f(x)$

(c) $\lim_{x \to -1} f(x)$

Finding a Limit **In Exercises 9–36, find the limit (if it exists). Use a graphing utility to confirm your result graphically.**

9. $\lim_{x \to 6} \dfrac{x - 6}{x^2 - 36}$

10. $\lim_{x \to 9} \dfrac{9 - x}{x^2 - 81}$

11. $\lim_{x \to 2} \dfrac{x^2 - x - 2}{x - 2}$

12. $\lim_{x \to -1} \dfrac{x^2 + 6x + 5}{x + 1}$

13. $\lim_{x \to -1} \dfrac{1 - 2x - 3x^2}{1 + x}$

14. $\lim_{x \to -4} \dfrac{2x^2 + 7x - 4}{x + 4}$

15. $\lim_{t \to 2} \dfrac{t^3 - 8}{t - 2}$

16. $\lim_{a \to -4} \dfrac{a^3 + 64}{a + 4}$

17. $\lim_{x \to 2} \dfrac{x^5 - 32}{x - 2}$

18. $\lim_{x \to 1} \dfrac{x^4 - 1}{x - 1}$

19. $\lim_{x \to -4} \dfrac{x^2 + x - 12}{x^2 + 6x + 8}$

20. $\lim_{x \to 3} \dfrac{x^2 - 8x + 15}{x^2 - 2x - 3}$

21. $\lim_{x \to -1} \dfrac{x^3 + 2x^2 - x - 2}{x^3 + 4x^2 - x - 4}$

22. $\lim_{x \to -3} \dfrac{x^3 + 2x^2 - 9x - 18}{x^3 + x^2 - 9x - 9}$

23. $\lim_{y \to 0} \dfrac{\sqrt{5 + y} - \sqrt{5}}{y}$

24. $\lim_{z \to 0} \dfrac{\sqrt{7 - z} - \sqrt{7}}{z}$

25. $\lim_{x \to -3} \dfrac{\sqrt{x + 7} - 2}{x + 3}$

26. $\lim_{x \to 2} \dfrac{4 - \sqrt{18 - x}}{x - 2}$

27. $\lim_{x \to 0} \dfrac{\dfrac{1}{x + 1} - 1}{x}$

28. $\lim_{x \to 0} \dfrac{\dfrac{1}{x - 8} + \dfrac{1}{8}}{x}$

29. $\lim_{x \to 0} \dfrac{\dfrac{1}{x + 4} - \dfrac{1}{4}}{x}$

30. $\lim_{x \to 0} \dfrac{\dfrac{1}{2 + x} - \dfrac{1}{2}}{x}$

31. $\lim_{x \to \pi/2} \dfrac{1 - \sin x}{\cos x}$

32. $\lim_{x \to 0} \dfrac{\cos x - 1}{\sin x}$

33. $\lim_{x \to 0} \dfrac{\cos 2x}{\cot 2x}$

34. $\lim_{x \to \pi} \dfrac{\sin x}{\csc x}$

35. $\lim_{x \to \pi/2} \dfrac{\sin x - 1}{x}$

36. $\lim_{x \to \pi} \dfrac{1 + \cos x}{x}$

Approximating a Limit Numerically **In Exercises 37–42, use the *table* feature of a graphing utility to create a table for the function and use the result to approximate the limit numerically. Write an approximation that is accurate to three decimal places.**

37. $\lim_{x \to 0} \frac{e^{2x} - 1}{x}$ **38.** $\lim_{x \to 0} \frac{1 - e^{-x}}{x}$

39. $\lim_{x \to 0} \frac{\sqrt{2x + 1} - 1}{x}$ **40.** $\lim_{x \to 9} \frac{3 - \sqrt{x}}{x - 9}$

41. $\lim_{x \to 0} (1 - x)^{2/x}$ **42.** $\lim_{x \to 0} (1 + 2x)^{1/x}$

Approximating a Limit Graphically **In Exercises 43–48, use a graphing utility to graph the function and approximate the limit. Write an approximation that is accurate to three decimal places.**

43. $\lim_{x \to 0} \frac{\sin 2x}{x}$ **44.** $\lim_{x \to 0} \frac{\sin 3x}{x}$

45. $\lim_{x \to 0} \frac{\tan x}{x}$ **46.** $\lim_{x \to 0} \frac{1 - \cos 2x}{x}$

47. $\lim_{x \to 1} \frac{1 - \sqrt[3]{x}}{1 - x}$ **48.** $\lim_{x \to 1} \frac{\sqrt[3]{x} - x}{x - 1}$

Evaluating One-Sided Limits **In Exercises 49–56, graph the function. Determine the limit (if it exists) by evaluating the corresponding one-sided limits.**

49. $\lim_{x \to 6} \frac{|x - 6|}{x - 6}$ **50.** $\lim_{x \to 2} \frac{|x - 2|}{x - 2}$

51. $\lim_{x \to 1} \frac{1}{x^2 + 1}$ **52.** $\lim_{x \to 1} \frac{1}{x^2 - 1}$

53. $\lim_{x \to 2} f(x)$ where $f(x) = \begin{cases} x - 1, & x \leq 2 \\ 2x - 3, & x > 2 \end{cases}$

54. $\lim_{x \to 1} f(x)$ where $f(x) = \begin{cases} 2x + 1, & x < 1 \\ 4 - x^2, & x \geq 1 \end{cases}$

55. $\lim_{x \to 1} f(x)$ where $f(x) = \begin{cases} 4 - x^2, & x \leq 1 \\ 3 - x, & x > 1 \end{cases}$

56. $\lim_{x \to 0} f(x)$ where $f(x) = \begin{cases} 4 - x^2, & x \leq 0 \\ x + 4, & x > 0 \end{cases}$

Algebraic-Graphical-Numerical **In Exercises 57–60, (a) graphically approximate the limit (if it exists) by using a graphing utility to graph the function, (b) numerically approximate the limit (if it exists) by using the *table* feature of the graphing utility to create a table, and (c) algebraically evaluate the limit (if it exists) by the appropriate technique(s).**

57. $\lim_{x \to 1^-} \frac{x - 1}{x^2 - 1}$ **58.** $\lim_{x \to 5^+} \frac{5 - x}{25 - x^2}$

59. $\lim_{x \to 16^+} \frac{4 - \sqrt{x}}{x - 16}$ **60.** $\lim_{x \to 0^-} \frac{\sqrt{x + 2} - \sqrt{2}}{x}$

Finding a Limit **In Exercises 61–66, use a graphing utility to graph the function and the equations $y = x$ and $y = -x$ in the same viewing window. Use the graph to find $\lim_{x \to 0} f(x)$.**

61. $f(x) = x \cos x$ **62.** $f(x) = |x \sin x|$

63. $f(x) = |x| \sin x$ **64.** $f(x) = |x| \cos x$

65. $f(x) = x \sin \frac{1}{x}$ **66.** $f(x) = x \cos \frac{1}{x}$

Finding Limits **In Exercises 67 and 68, state which limit can be evaluated by using direct substitution. Then evaluate or approximate each limit.**

67. (a) $\lim_{x \to 0} x^2 \sin x^2$ (b) $\lim_{x \to 0} \frac{\sin x^2}{x^2}$

68. (a) $\lim_{x \to 0} \frac{x}{\cos x}$ (b) $\lim_{x \to 0} \frac{1 - \cos x}{x}$

Why you should learn it *(p. 760)* **In Exercises 69 and 70, use the position function**

$$s(t) = -16t^2 + 128$$

which gives the height (in feet) of a free-falling object. The velocity at time $t = a$ seconds is given by

$$\lim_{t \to a} \frac{s(a) - s(t)}{a - t}.$$

69. Find the velocity when $t = 1$ second.

70. Find the velocity when $t = 2$ seconds.

71. Human Resources A union contract guarantees an 8% salary increase yearly for 3 years. For a current salary of \$30,000, the salaries $f(t)$ (in thousands of dollars) for the next 3 years are given by

$$f(t) = \begin{cases} 30.000, & 0 < t \leq 1 \\ 32.400, & 1 < t \leq 2 \\ 34.992, & 2 < t \leq 3 \end{cases}$$

where t represents the time in years. Show that the limit of f as $t \to 2$ does not exist.

72. Business The cost of sending a package overnight is \$15 for the first pound and \$1.30 for each additional pound or portion of a pound. A plastic mailing bag can hold up to 3 pounds. The cost $f(x)$ of sending a package in a plastic mailing bag is given by

$$f(x) = \begin{cases} 15.00, & 0 < x \leq 1 \\ 16.30, & 1 < x \leq 2 \\ 17.60, & 2 < x \leq 3 \end{cases}$$

where x represents the weight of the package (in pounds). Show that the limit of f as $x \to 1$ does not exist.

73. MODELING DATA

The cost of hooking up and towing a car is $85 for the first mile and $5 for each additional mile or portion of a mile. A model for the cost C (in dollars) is $C(x) = 85 - 5[\![-(x - 1)]\!]$, where x is the distance in miles. (Recall from Section 1.3 that $f(x) = [\![x]\!] =$ the greatest integer less than or equal to x.)

(a) Use a graphing utility to graph C for $0 < x \le 10$.

(b) Complete the table and observe the behavior of C as x approaches 5.5. Use the graph from part (a) and the table to find $\lim_{x \to 5.5} C(x)$.

x	5	5.3	5.4	5.5	5.6	5.7	6
$C(x)$				?			

(c) Complete the table and observe the behavior of C as x approaches 5. Does the limit of $C(x)$ as x approaches 5 exist? Explain.

x	4	4.5	4.9	5	5.1	5.5	6
$C(x)$				?			

74. MODELING DATA

The cost C (in dollars) of making x photocopies at a copy shop is given by the function

$$C(x) = \begin{cases} 0.15x, & 0 < x \le 25 \\ 0.10x, & 25 < x \le 100 \\ 0.07x, & 100 < x \le 500 \\ 0.05x, & x > 500 \end{cases}.$$

(a) Sketch a graph of the function.

(b) Find each limit and interpret your result in the context of the situation.

(i) $\lim_{x \to 15} C(x)$ (ii) $\lim_{x \to 99} C(x)$ (iii) $\lim_{x \to 305} C(x)$

(c) Create a table of values to show numerically that each limit does not exist.

(i) $\lim_{x \to 25} C(x)$ (ii) $\lim_{x \to 100} C(x)$ (iii) $\lim_{x \to 500} C(x)$

(d) Explain how you can use the graph in part (a) to verify that the limits in part (c) do not exist.

Evaluating a Limit from Calculus **In Exercises 75–82, find**

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

75. $f(x) = 3x - 1$ **76.** $f(x) = 5 - 6x$

77. $f(x) = \sqrt{x}$ **78.** $f(x) = \sqrt{x - 2}$

✓ **79.** $f(x) = x^2 - 3x$ **80.** $f(x) = 4 - 2x - x^2$

81. $f(x) = \dfrac{1}{x + 2}$ **82.** $f(x) = \dfrac{1}{x - 1}$

Conclusions

True or False? **In Exercises 83 and 84, determine whether the statement is true or false. Justify your answer.**

83. When your attempt to find the limit of a rational function yields the indeterminate form $\frac{0}{0}$, the rational function's numerator and denominator have a common factor.

84. If $f(c) = L$, then $\lim_{x \to c} f(x) = L$.

85. Think About It Sketch the graph of a function for which $f(2)$ is defined but the limit of $f(x)$ as x approaches 2 does not exist.

86. Think About It Sketch the graph of a function for which the limit of $f(x)$ as x approaches 1 is 4 but $f(1) \neq 4$.

87. Writing Consider the limit of the rational function $p(x)/q(x)$. What conclusion can you make when direct substitution produces each expression? Write a short paragraph explaining your reasoning.

(a) $\lim_{x \to c} \dfrac{p(x)}{q(x)} = \dfrac{0}{1}$ (b) $\lim_{x \to c} \dfrac{p(x)}{q(x)} = \dfrac{1}{1}$

(c) $\lim_{x \to c} \dfrac{p(x)}{q(x)} = \dfrac{1}{0}$ (d) $\lim_{x \to c} \dfrac{p(x)}{q(x)} = \dfrac{0}{0}$

88. CAPSTONE Given

$$f(x) = \begin{cases} 2x, & x \le 0 \\ x^2 + 1, & x > 0 \end{cases},$$

find each of the following limits. If the limit does not exist, explain why.

(a) $\lim_{x \to 0^-} f(x)$ (b) $\lim_{x \to 0^+} f(x)$ (c) $\lim_{x \to 0} f(x)$

Cumulative Mixed Review

Identifying a Conic from Its Equation **In Exercises 89–92, identify the type of conic represented by the equation. Use a graphing utility to confirm your result.**

89. $r = \dfrac{3}{1 + \cos\theta}$ **90.** $r = \dfrac{12}{3 + 2\sin\theta}$

91. $r = \dfrac{9}{2 + 3\cos\theta}$ **92.** $r = \dfrac{4}{4 + \cos\theta}$

A Relationship of Two Vectors **In Exercises 93–96, determine whether the vectors are orthogonal, parallel, or neither.**

93. $\langle 7, -2, 3\rangle, \langle -1, 4, 5\rangle$ **94.** $\langle 5, 5, 0\rangle, \langle 0, 5, 1\rangle$

95. $\langle 2, -3, 1\rangle, \langle -2, 2, 2\rangle$ **96.** $\langle -1, 3, 1\rangle, \langle 3, -9, -3\rangle$

11.3 The Tangent Line Problem

Tangent Line to a Graph

Calculus is a branch of mathematics that studies rates of change of functions. If you go on to take a course in calculus, you will learn that rates of change have many applications in real life.

Earlier in the text, you learned how the slope of a line indicates the rate at which a line rises or falls. For a line, this rate (or slope) is the same at every point on the line. For graphs other than lines, the rate at which the graph rises or falls changes from point to point. For instance, in Figure 11.18, the parabola is rising more quickly at the point (x_1, y_1) than it is at the point (x_2, y_2). At the vertex (x_3, y_3), the graph levels off, and at the point (x_4, y_4), the graph is falling.

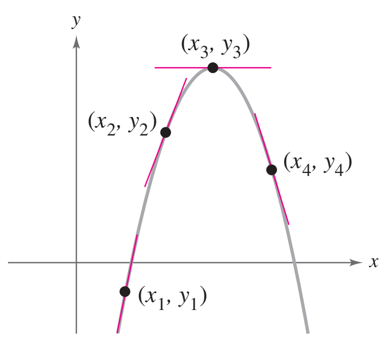


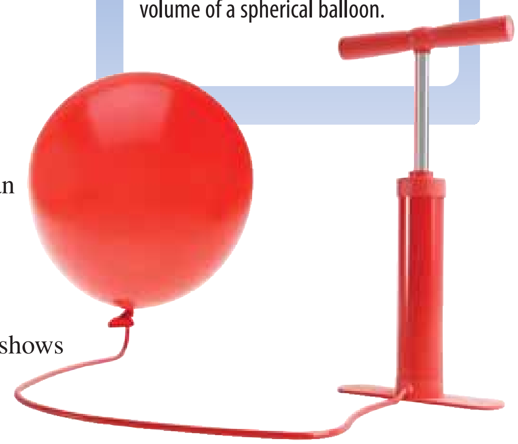
Figure 11.18

What you should learn

- Understand the tangent line problem.
- Use a tangent line to approximate the slope of a graph at a point.
- Use the limit definition of slope to find exact slopes of graphs.
- Find derivatives of functions and use derivatives to find slopes of graphs.

Why you should learn it

The derivative, or the slope of the tangent line to the graph of a function at a point, can be used to analyze rates of change. For instance, in Exercise 69 on page 779, the derivative is used to analyze the rate of change of the volume of a spherical balloon.



To determine the rate at which a graph rises or falls at a *single point*, you can find the slope of the tangent line at that point. In simple terms, the **tangent line** to the graph of a function f at a point

$$P(x_1, y_1)$$

is the line that best approximates the slope of the graph at the point. Figure 11.19 shows other examples of tangent lines.

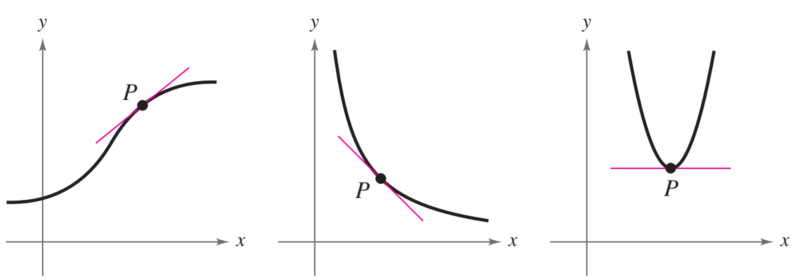


Figure 11.19

From geometry, you know that a line is tangent to a circle when the line intersects the circle at only one point (see Figure 11.20). Tangent lines to noncircular graphs, however, can intersect the graph at more than one point. For instance, in the first graph in Figure 11.19, if the tangent line were extended, then it would intersect the graph at a point other than the point of tangency.

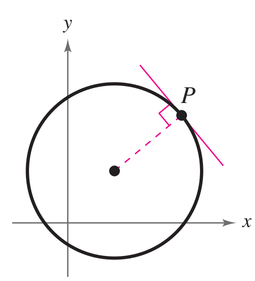


Figure 11.20

Slope of a Graph

Because a tangent line approximates the slope of a graph at a point, the problem of finding the slope of a graph at a point is the same as finding the slope of the tangent line at the point.

Example 1 Visually Approximating the Slope of a Graph

Use the graph in Figure 11.21 to approximate the slope of the graph of

$$f(x) = x^2$$

at the point $(1, 1)$.

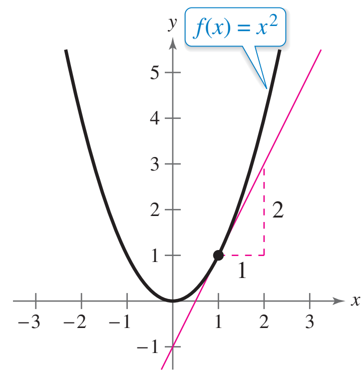


Figure 11.21

Solution

From the graph of $f(x) = x^2$, you can see that the tangent line at $(1, 1)$ rises approximately two units for each unit change in x. So, you can estimate the slope of the tangent line at $(1, 1)$ to be

$$\text{Slope} = \frac{\text{change in } y}{\text{change in } x}$$

$$\approx \frac{2}{1}$$

$$= 2.$$

Because the tangent line at the point $(1, 1)$ has a slope of about 2, you can conclude that the graph of f has a slope of about 2 at the point $(1, 1)$.

CHECKPOINT Now try Exercise 7.

When you are visually approximating the slope of a graph, remember that the scales on the horizontal and vertical axes may differ. When this happens (as it frequently does in applications), the slope of the tangent line is distorted, and you must be careful to account for the difference in the scales.

Example 2 Approximating the Slope of a Graph

Figure 11.22 graphically depicts the monthly normal temperatures (in degrees Fahrenheit) for Dallas, Texas. Approximate the slope of this graph at the indicated point and give a physical interpretation of the result. (Source: National Climatic Data Center)

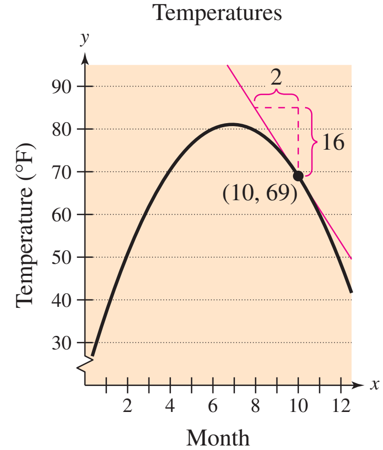


Figure 11.22

Solution

From the graph, you can see that the tangent line at the given point falls approximately 16 units for each two-unit change in x. So, you can estimate the slope at the given point to be

$$\text{Slope} = \frac{\text{change in } y}{\text{change in } x}$$

$$\approx \frac{-16}{2}$$

$$= -8 \text{ degrees per month.}$$

This means that you can expect the monthly normal temperature in November to be about 8 degrees lower than the normal temperature in October.

CHECKPOINT Now try Exercise 9.

Slope and the Limit Process

In Examples 1 and 2, you approximated the slope of a graph at a point by creating a graph and then "eyeballing" the tangent line at the point of tangency. A more systematic method of approximating tangent lines makes use of a **secant line** through the point of tangency and a second point on the graph, as shown in Figure 11.23. If $(x, f(x))$ is the point of tangency and

$$(x + h, f(x + h))$$

is a second point on the graph of f, then the slope of the secant line through the two points is given by

$$m_{\text{sec}} = \frac{f(x + h) - f(x)}{h}. \qquad \text{Slope of secant line}$$

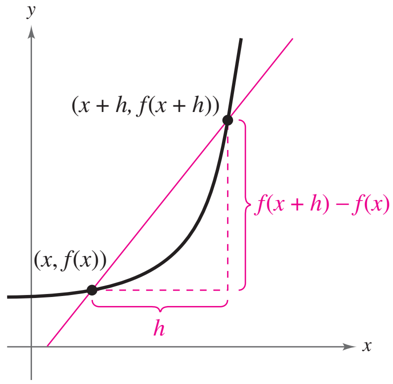
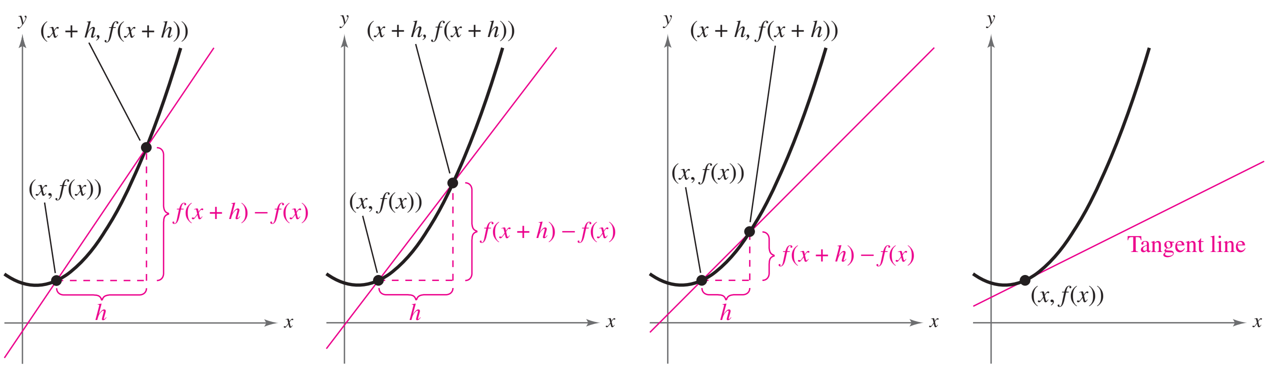


Figure 11.23

The right side of this equation is called the **difference quotient.** The denominator h is the *change in* x, and the numerator is the *change in* y. The beauty of this procedure is that you obtain more and more accurate approximations of the slope of the tangent line by choosing points closer and closer to the point of tangency, as shown in Figure 11.24.



As h approaches 0, the secant line approaches the tangent line.
Figure 11.24

Using the limit process, you can find the *exact* slope of the tangent line at $(x, f(x))$.

Definition of the Slope of a Graph

The **slope** m of the graph of f at the point $(x, f(x))$ is equal to the slope of its tangent line at $(x, f(x))$, and is given by

$$\begin{aligned} m &= \lim_{h \to 0} m_{\text{sec}} \\ &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \end{aligned}$$

provided this limit exists.

From the definition above and from Section 11.2, you can see that the difference quotient is used frequently in calculus. Using the difference quotient to find the slope of a tangent line to a graph is a major concept of calculus.

Example 3 Finding the Slope of a Graph

Find the slope of the graph of $f(x) = x^2$ at the point $(-2, 4)$.

Solution

Find an expression that represents the slope of a secant line at $(-2, 4)$.

$$m_{\text{sec}} = \frac{f(-2+h) - f(-2)}{h} \qquad \text{Set up difference quotient.}$$

$$= \frac{(-2+h)^2 - (-2)^2}{h} \qquad \text{Substitute into } f(x) = x^2.$$

$$= \frac{4 - 4h + h^2 - 4}{h} \qquad \text{Expand terms.}$$

$$= \frac{-4h + h^2}{h} \qquad \text{Simplify.}$$

$$= \frac{\cancel{h}(-4+h)}{\cancel{h}} \qquad \text{Factor and divide out.}$$

$$= -4 + h, \quad h \neq 0 \qquad \text{Simplify.}$$

Next, take the limit of m_{sec} as h approaches 0.

$$m = \lim_{h \to 0} m_{\text{sec}}$$
$$= \lim_{h \to 0} (-4 + h)$$
$$= -4 + 0$$
$$= -4$$

The graph has a slope of -4 at the point $(-2, 4)$, as shown in Figure 11.25.

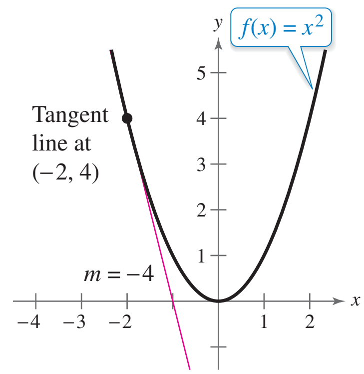


Figure 11.25

CHECKPOINT Now try Exercise 11.

Example 4 Finding the Slope of a Graph

Find the slope of $f(x) = -2x + 4$.

Solution

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \text{Set up difference quotient.}$$

$$= \lim_{h \to 0} \frac{[-2(x+h) + 4] - (-2x + 4)}{h} \qquad \text{Substitute into } f(x) = -2x + 4.$$

$$= \lim_{h \to 0} \frac{-2x - 2h + 4 + 2x - 4}{h} \qquad \text{Expand terms.}$$

$$= \lim_{h \to 0} \frac{-2\cancel{h}}{\cancel{h}} \qquad \text{Divide out.}$$

$$= -2 \qquad \text{Simplify.}$$

You know from your study of linear functions that the line given by

$$f(x) = -2x + 4$$

has a slope of -2, as shown in Figure 11.26. This conclusion is consistent with that obtained by the limit definition of slope, as shown above.

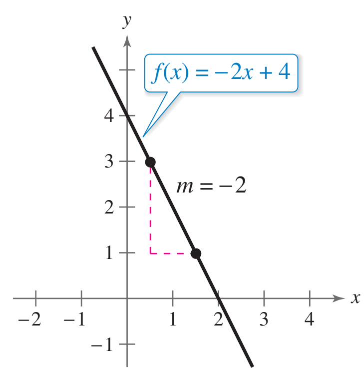


Figure 11.26

CHECKPOINT Now try Exercise 13.

It is important that you see the difference between the ways the difference quotients were set up in Examples 3 and 4. In Example 3, you were finding the slope of a graph at a specific point $(c, f(c))$. To find the slope in such a case, you can use the following form of the difference quotient.

$$m = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \qquad \text{Slope at specific point}$$

In Example 4, however, you were finding a *formula* for the slope at *any* point on the graph. In such cases, you should use x, rather than c, in the difference quotient.

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \text{Formula for slope}$$

Example 5 Finding a Formula for the Slope of a Graph

Find a formula for the slope of the graph of

$f(x) = x^2 + 1.$

What are the slopes at the points $(-1, 2)$ and $(2, 5)$?

Solution

$$m_{sec} = \frac{f(x+h) - f(x)}{h} \qquad \text{Set up difference quotient.}$$

$$= \frac{[(x+h)^2 + 1] - (x^2 + 1)}{h} \qquad \text{Substitute into } f(x) = x^2 + 1.$$

$$= \frac{x^2 + 2xh + h^2 + 1 - x^2 - 1}{h} \qquad \text{Expand terms.}$$

$$= \frac{2xh + h^2}{h} \qquad \text{Simplify.}$$

$$= \frac{\cancel{h}(2x + h)}{\cancel{h}} \qquad \text{Factor and divide out.}$$

$$= 2x + h, \quad h \neq 0 \qquad \text{Simplify.}$$

Next, take the limit of m_{sec} as h approaches 0.

$$m = \lim_{h \to 0} m_{sec}$$
$$= \lim_{h \to 0} (2x + h)$$
$$= 2x + 0$$
$$= 2x$$

Using the formula $m = 2x$ for the slope at $(x, f(x))$, you can find the slope at the specified points. At $(-1, 2)$, the slope is

$m = 2(-1) = -2$

and at $(2, 5)$, the slope is

$m = 2(2) = 4.$

The graph of f is shown in Figure 11.27.

CHECKPOINT Now try Exercise 19.

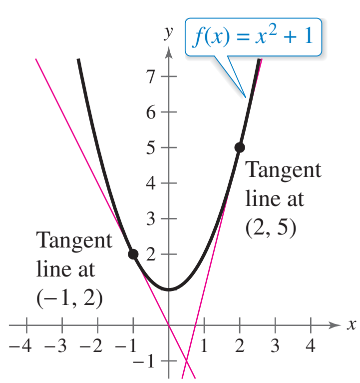


Figure 11.27

Technology Tip

Try verifying the result in Example 5 by using a graphing utility to graph the function and the tangent lines at $(-1, 2)$ and $(2, 5)$ as

$y_1 = x^2 + 1$

$y_2 = -2x$

$y_3 = 4x - 3$

in the same viewing window. You can also verify the result using the *tangent* feature. For instructions on how to use the *tangent* feature, see Appendix A; for specific keystrokes, go to this textbook's *Companion Website.*

The Derivative of a Function

In Example 5, you started with the function $f(x) = x^2 + 1$ and used the limit process to derive another function, $m = 2x$, that represents the slope of the graph of f at the point $(x, f(x))$. This derived function is called the **derivative** of f at x. It is denoted by $f'(x)$, which is read as "f prime of x."

> **Definition of the Derivative**
>
> The **derivative** of f at x is given by
>
> $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
>
> provided this limit exists.

Remember that the derivative $f'(x)$ is a formula for the slope of the tangent line to the graph of f at the point $(x, f(x))$.

Example 6 Finding a Derivative

algebra of calculus

Find the derivative of

$$f(x) = 3x^2 - 2x.$$

Solution

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{[3(x+h)^2 - 2(x+h)] - (3x^2 - 2x)}{h} \\
&= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 2x - 2h - 3x^2 + 2x}{h} \\
&= \lim_{h \to 0} \frac{6xh + 3h^2 - 2h}{h} \\
&= \lim_{h \to 0} \frac{\cancel{h}(6x + 3h - 2)}{\cancel{h}} \\
&= \lim_{h \to 0} (6x + 3h - 2) \\
&= 6x + 3(0) - 2 \\
&= 6x - 2
\end{aligned}$$

So, the derivative of $f(x) = 3x^2 - 2x$ is

$$f'(x) = 6x - 2. \qquad \text{Derivative of } f \text{ at } x$$

CHECKPOINT Now try Exercise 31.

Note that in addition to $f'(x)$, other notations can be used to denote the derivative of $y = f(x)$. The most common are

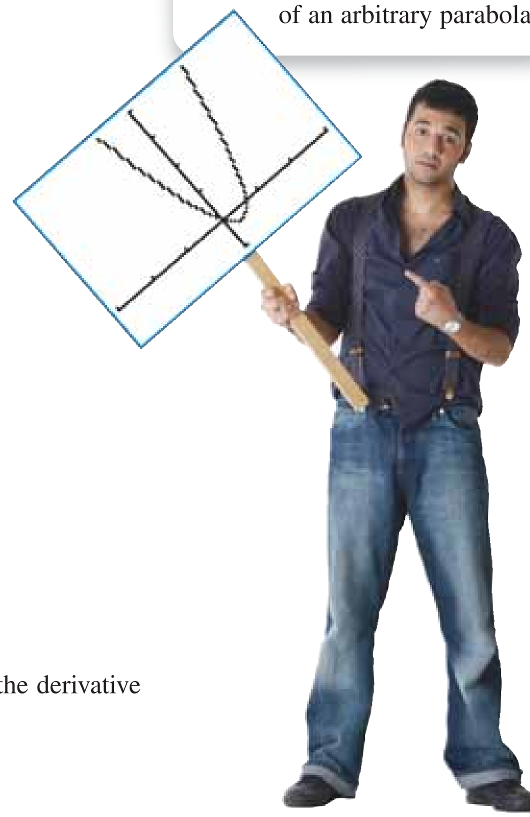
$$\frac{dy}{dx}, \quad y', \quad \frac{d}{dx}[f(x)], \quad \text{and} \quad D_x[y].$$

Study Tip

In Section 1.1, you studied the slope of a line, which represents the *average rate of change* over an interval. The derivative of a function is a formula which represents the *instantaneous rate of change* at a point.

Explore the Concept

Use a graphing utility to graph the function $f(x) = 3x^2 - 2x$. Use the *trace* feature to approximate the coordinates of the vertex of this parabola. Then use the derivative of $f(x) = 3x^2 - 2x$ to find the slope of the tangent line at the vertex. Make a conjecture about the slope of the tangent line at the vertex of an arbitrary parabola.



Hasan Kursad Ergan/iStockphoto.com

Example 7 Using the Derivative

Find $f'(x)$ for

$$f(x) = \sqrt{x}.$$

Then find the slopes of the graph of f at the points $(1, 1)$ and $(4, 2)$ and equations of the tangent lines to the graph at the points.

Additional Example
Find the derivative of

$f(x) = x^2 - 5x.$

Answer: $f'(x) = 2x - 5$

Solution

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

Because direct substitution yields the indeterminate form $\frac{0}{0}$, you should use the rationalizing technique discussed in Section 11.2 to find the limit.

$$f'(x) = \lim_{h\to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h}\right)\left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right)$$

$$= \lim_{h\to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x+0} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

Study Tip

Remember that in order to rationalize the numerator of an expression, you must multiply the numerator and denominator by the conjugate of the numerator.

At the point $(1, 1)$, the slope is

$$f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}.$$

Activity
Ask your students to graph $f(t) = 3/t$ and identify the point $(3, 1)$ on the graph to give some meaning to the task of finding the slope at that point. You might also consider asking your students to find this limit numerically, for the sake of comparison.

An equation of the tangent line at the point $(1, 1)$ is

$$y - y_1 = m(x - x_1) \qquad \text{Point-slope form}$$

$$y - 1 = \tfrac{1}{2}(x - 1) \qquad \text{Substitute } \tfrac{1}{2} \text{ for } m, 1 \text{ for } x_1, \text{ and } 1 \text{ for } y_1.$$

$$y = \tfrac{1}{2}x + \tfrac{1}{2}. \qquad \text{Tangent line}$$

At the point $(4, 2)$, the slope is

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

An equation of the tangent line at the point $(4, 2)$ is

$$y - y_1 = m(x - x_1) \qquad \text{Point-slope form}$$

$$y - 2 = \tfrac{1}{4}(x - 4) \qquad \text{Substitute } \tfrac{1}{4} \text{ for } m, 4 \text{ for } x_1, \text{ and } 2 \text{ for } y_1.$$

$$y = \tfrac{1}{4}x + 1. \qquad \text{Tangent line}$$

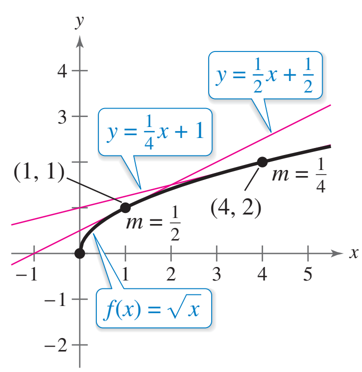


Figure 11.28

The graphs of f and the tangent lines at the points $(1, 1)$ and $(4, 2)$ are shown in Figure 11.28.

CHECKPOINT Now try Exercise 43.

11.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1–4, fill in the blank.

1. _______ is the study of the rates of change of functions.

2. The _______ to the graph of a function at a point is the line that best approximates the slope of the graph at the point.

3. To approximate a tangent line to a graph, you can make use of a _______ through the point of tangency and a second point on the graph.

4. The _______ of a function f at x represents the slope of the graph of f at the point $(x, f(x))$.

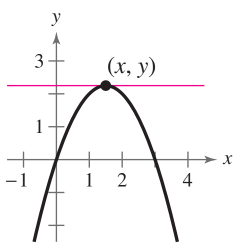
5. The slope of the tangent line to the graph of f at the point $(1, 5)$ is 2. What is the slope of the graph of f at the point $(1, 5)$?

6. Given $f(1) = 2$ and $f'(1) = -4$, what is the slope of the graph of f at the point $(1, 2)$?

Procedures and Problem Solving

Approximating the Slope of a Graph **In Exercises 7–10, use the figure to approximate the slope of the curve at the point (x, y).**

7.



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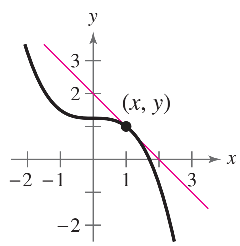

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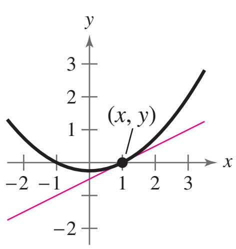

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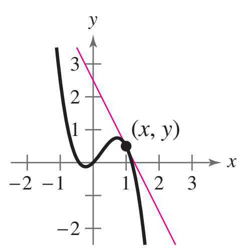


Finding the Slope of a Graph **In Exercises 11–18, use the limit process to find the slope of the graph of the function at the specified point. Use a graphing utility to confirm your result.**

11. $g(x) = x^2 - 4x$, $(3, -3)$

12. $f(x) = 10x - 2x^2$, $(3, 12)$

13. $g(x) = 5 - 2x$, $(1, 3)$

14. $h(x) = 2x + 5$, $(-1, -3)$

15. $g(x) = \dfrac{4}{x}$, $(2, 2)$

16. $g(x) = \dfrac{1}{x - 2}$, $\left(4, \dfrac{1}{2}\right)$

17. $h(x) = \sqrt{x}$, $(9, 3)$

18. $h(x) = \sqrt{x + 10}$, $(-1, 3)$

Finding a Formula for the Slope of a Graph **In Exercises 19–24, find a formula for the slope of the graph of f at the point $(x, f(x))$. Then use it to find the slopes at the two specified points.**

19. $f(x) = 4 - x^2$
(a) $(0, 4)$
(b) $(-2, 0)$

20. $f(x) = x^3$
(a) $(1, 1)$
(b) $(2, 8)$

21. $f(x) = \dfrac{1}{x + 4}$
(a) $\left(0, \frac{1}{4}\right)$
(b) $\left(-2, \frac{1}{2}\right)$

22. $f(x) = \dfrac{1}{x + 2}$
(a) $\left(0, \frac{1}{2}\right)$
(b) $(-3, -1)$

23. $f(x) = \sqrt{x - 1}$
(a) $(2, 1)$
(b) $(10, 3)$

24. $f(x) = \sqrt{x - 4}$
(a) $(5, 1)$
(b) $(8, 2)$

Approximating the Slope of a Tangent Line **In Exercises 25–30, use a graphing utility to graph the function and the tangent line at the point $(1, f(1))$. Use the graph to approximate the slope of the tangent line.**

25. $f(x) = x^2 - 3$

26. $f(x) = x^2 - 2x + 1$

27. $f(x) = \sqrt{2 - x}$

28. $f(x) = \sqrt{x + 3}$

29. $f(x) = \dfrac{4}{x + 1}$

30. $f(x) = \dfrac{3}{2 - x}$

Finding a Derivative **In Exercises 31–42, find the derivative of the function.**

31. $f(x) = 4 - 3x^2$

32. $f(x) = x^2 - 3x + 4$

33. $f(x) = 5$

34. $f(x) = -1$

35. $f(x) = 9 - \frac{1}{3}x$

36. $f(x) = -5x + 2$

37. $f(x) = \dfrac{1}{x^2}$

38. $f(x) = \dfrac{1}{x^3}$

39. $f(x) = \sqrt{x - 4}$

40. $f(x) = \sqrt{x + 1}$

41. $f(x) = \dfrac{1}{\sqrt{x-9}}$ **42.** $f(x) = \dfrac{1}{\sqrt{x+1}}$

Using the Derivative **In Exercises 43–50, (a) find the slope of the graph of f at the given point, (b) find an equation of the tangent line to the graph at the point, and (c) graph the function and the tangent line.**

✓ **43.** $f(x) = x^2 - 1, (2, 3)$ **44.** $f(x) = 4 - x^2, (1, 3)$

45. $f(x) = x^3 - 2x, (1, -1)$

46. $f(x) = x^2 + 2x + 1, (-3, 4)$

47. $f(x) = \sqrt{x+1}, (3, 2)$ **48.** $f(x) = \sqrt{x-2}, (3, 1)$

49. $f(x) = \dfrac{1}{x+5}, (-4, 1)$ **50.** $f(x) = \dfrac{1}{x-3}, (4, 1)$

Graphing a Function Over an Interval **In Exercises 51–54, use a graphing utility to graph f over the interval $[-2, 2]$ and complete the table. Compare the value of the first derivative with a visual approximation of the slope of the graph.**

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
$f(x)$									
$f'(x)$									

51. $f(x) = \frac{1}{2}x^2$ **52.** $f(x) = \frac{1}{4}x^3$

53. $f(x) = \sqrt{x+3}$ **54.** $f(x) = \dfrac{x^2-4}{x+4}$

Using the Derivative **In Exercises 55–58, find the derivative of f. Use the derivative to determine any points on the graph of f at which the tangent line is horizontal. Use a graphing utility to verify your results.**

55. $f(x) = x^2 - 4x + 3$ **56.** $f(x) = x^2 - 6x + 4$

57. $f(x) = 3x^3 - 9x$ **58.** $f(x) = x^3 + 3x$

Using the Derivative **In Exercises 59–66, use the function and its derivative to determine any points on the graph of f at which the tangent line is horizontal. Use a graphing utility to verify your results.**

59. $f(x) = x^4 - 2x^2, f'(x) = 4x^3 - 4x$

60. $f(x) = 3x^4 + 4x^3, f'(x) = 12x^3 + 12x^2$

61. $f(x) = 2\cos x + x, f'(x) = -2\sin x + 1$, over the interval $(0, 2\pi)$

62. $f(x) = x - 2\sin x, f'(x) = 1 - 2\cos x$, over the interval $(0, 2\pi)$

63. $f(x) = x^2e^x, f'(x) = x^2e^x + 2xe^x$

64. $f(x) = xe^{-x}, f'(x) = e^{-x} - xe^{-x}$

65. $f(x) = x\ln x, f'(x) = \ln x + 1$

66. $f(x) = \dfrac{\ln x}{x}, f'(x) = \dfrac{1 - \ln x}{x^2}$

67. MODELING DATA

The projected populations y (in thousands) of New Jersey for selected years from 2015 to 2030 are shown in the table. (Source: U.S. Census Bureau)

Year	Population, y (in thousands)
2015	9256
2020	9462
2025	9637
2030	9802

(a) Use the *regression* feature of a graphing utility to find a quadratic model for the data. Let t represent the year, with $t = 15$ corresponding to 2015.

(b) Use the graphing utility to graph the model found in part (a). Estimate the slope of the graph when $t = 20$, and interpret the result.

(c) Find the derivative of the model in part (a). Then evaluate the derivative for $t = 20$.

(d) Write a brief statement regarding your results for parts (a) through (c).

68. MODELING DATA

The data in the table show the number N (in thousands) of books sold when the price per book is p (in dollars).

Price, p	Number of books, N (in thousands)
\$10	900
\$15	630
\$20	396
\$25	227
\$30	102
\$35	36

(a) Use the *regression* feature of a graphing utility to find a quadratic model for the data.

(b) Use the graphing utility to graph the model found in part (a). Estimate the slopes of the graph when $p = \$15$ and $p = \$30$.

(c) Use the graphing utility to graph the tangent lines to the model when $p = \$15$ and $p = \$30$. Compare the slopes given by the graphing utility with your estimates in part (b).

(d) The slopes of the tangent lines at $p = \$15$ and $p = \$30$ are not the same. Explain what this means to the company selling the books.

69. ***Why you should learn it*** *(p. 770)* A spherical balloon is inflated. The volume V is approximated by the formula $V(r) = \frac{4}{3}\pi r^3$, where r is the radius.

(a) Find the derivative of V with respect to r.

(b) Evaluate the derivative when the radius is 4 inches.

(c) What type of unit would be applied to your answer in part (b)? Explain.

70. **Rate of Change** An approximately spherical benign tumor is reducing in size. The surface area S is given by the formula $S(r) = 4\pi r^2$, where r is the radius.

(a) Find the derivative of S with respect to r.

(b) Evaluate the derivative when the radius is 3 millimeters.

(c) What type of unit would be applied to your answer in part (b)? Explain.

71. **Vertical Motion** A water balloon is thrown upward from the top of an 80-foot building with a velocity of 64 feet per second. The height or displacement s (in feet) of the balloon can be modeled by the position function $s(t) = -16t^2 + 64t + 80$, where t is the time in seconds from when it was thrown.

(a) Find a formula for the instantaneous rate of change of the balloon.

(b) Find the average rate of change of the balloon after the first three seconds of flight. Explain your results.

(c) Find the time at which the balloon reaches its maximum height. Explain your method.

(d) Velocity is given by the derivative of the position function. Find the velocity of the balloon as it impacts the ground.

(e) Use a graphing utility to graph the model and verify your results for parts (a)–(d).

72. **Vertical Motion** A coin is dropped from the top of a 120-foot building. The height or displacement s (in feet) of the coin can be modeled by the position function $s(t) = -16t^2 + 120$, where t is the time in seconds from when it was dropped.

(a) Find a formula for the instantaneous rate of change of the coin.

(b) Find the average rate of change of the coin after the first two seconds of free fall. Explain your results.

(c) Velocity is given by the derivative of the position function. Find the velocity of the coin as it impacts the ground.

(d) Find the time when the coin's velocity is -70 feet per second.

(e) Use a graphing utility to graph the model and verify your results for parts (a)–(d).

Conclusions

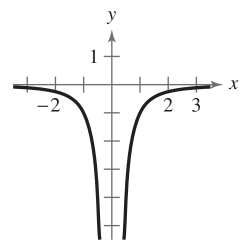
True or False? **In Exercises 73 and 74, determine whether the statement is true or false. Justify your answer.**

73. The slope of the graph of $y = x^2$ is different at every point on the graph of f.

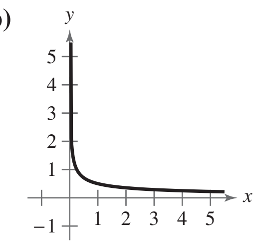
74. A tangent line to a graph can intersect the graph only at the point of tangency.

Graphing the Derivative of a Function **In Exercises 75–78, match the function with the graph of its *derivative*. It is not necessary to find the derivative of the function. [The graphs are labeled (a), (b), (c), and (d).]**

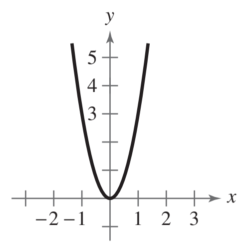
(a)



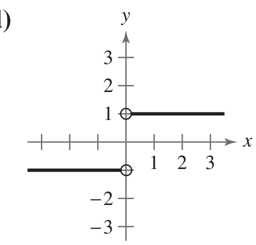
(b)



(c)



(d)



75. $f(x) = \sqrt{x}$

76. $f(x) = \dfrac{1}{x}$

77. $f(x) = |x|$

78. $f(x) = x^3$

79. **Think About It** Sketch the graph of a function whose derivative is always positive.

80. **CAPSTONE** Consider the graph of a function f.

(a) Explain how you can use a secant line to approximate the tangent line at $(x, f(x))$.

(b) Explain how you can use the limit process to find the exact slope of the tangent line at $(x, f(x))$.

81. **Think About It** Sketch the graph of a function for which $f'(x) < 0$ for $x < 1$, $f'(x) \geq 0$ for $x > 1$, and $f'(1) = 0$.

Cumulative Mixed Review

Sketching the Graph of a Rational Function **In Exercises 82 and 83, sketch the graph of the rational function.**

82. $f(x) = \dfrac{1}{x^2 - x - 2}$

83. $f(x) = \dfrac{x - 2}{x^2 - 4x + 3}$

11.4 Limits at Infinity and Limits of Sequences

Limits at Infinity and Horizontal Asymptotes

As pointed out at the beginning of this chapter, there are two basic problems in calculus: finding **tangent lines** and finding the **area** of a region. In Section 11.3, you saw how limits can be used to solve the tangent line problem. In this section and the next, you will see how a different type of limit, a *limit at infinity*, can be used to solve the area problem. To get an idea of what is meant by a limit at infinity, consider the function

$f(x) = (x + 1)/(2x)$.

The graph of f is shown in Figure 11.29. From earlier work, you know that $y = \frac{1}{2}$ is a horizontal asymptote of the graph of this function. Using limit notation, this can be written as follows.

$$\lim_{x \to -\infty} f(x) = \frac{1}{2} \qquad \text{Horizontal asymptote to the left}$$

$$\lim_{x \to \infty} f(x) = \frac{1}{2} \qquad \text{Horizontal asymptote to the right}$$

These limits mean that the value of $f(x)$ gets arbitrarily close to $\frac{1}{2}$ as x decreases or increases without bound.

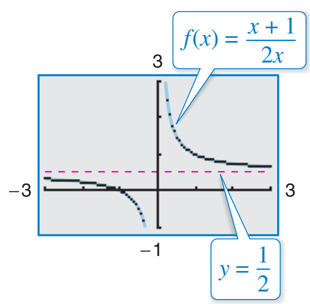


Figure 11.29

Definition of Limits at Infinity

If f is a function and L_1 and L_2 are real numbers, then the statements

$$\lim_{x \to -\infty} f(x) = L_1 \qquad \text{Limit as } x \text{ approaches } -\infty$$

and

$$\lim_{x \to \infty} f(x) = L_2 \qquad \text{Limit as } x \text{ approaches } \infty$$

denote the **limits at infinity.** The first statement is read "*the limit of $f(x)$ as x approaches $-\infty$ is L_1*," and the second is read "*the limit of $f(x)$ as x approaches ∞ is L_2*."

What you should learn

- Evaluate limits of functions at infinity.
- Find limits of sequences

Why you should learn it

Finding limits at infinity is useful in analyzing functions that model real-life situations. For instance, in Exercise 60 on page 788, you are asked to find a limit at infinity to decide whether you can use a given model to predict the payroll of the legislative branch of the United States government.

Technology Tip

Recall from Section 2.7 that some graphing utilities have difficulty graphing rational functions. In this text, rational functions are graphed using the *dot* mode of a graphing utility, and a blue curve is placed behind the graphing utility's display to indicate where the graph should appear.

To help evaluate limits at infinity, you can use the following definition.

Limits at Infinity

If r is a positive real number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0. \qquad \text{Limit toward the right}$$

Furthermore, if x^r is defined when $x < 0$, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0. \qquad \text{Limit toward the left}$$

Explore the Concept

Use a graphing utility to graph the two functions given by

$$y_1 = \frac{1}{\sqrt{x}} \quad \text{and} \quad y_2 = \frac{1}{\sqrt[3]{x}}$$

in the same viewing window. Why doesn't y_1 appear to the left of the y-axis? How does this relate to the statement at the left about the infinite limit

$$\lim_{x \to -\infty} \frac{1}{x^r}?$$

Limits at infinity share many of the properties of limits listed in Section 11.1. Some of these properties are demonstrated in the next example.

Example 1 Evaluating a Limit at Infinity

Find the limit.

$$\lim_{x \to \infty} \left(4 - \frac{3}{x^2}\right)$$

Algebraic Solution

Use the properties of limits listed in Section 11.1.

$$\begin{aligned}
\lim_{x \to \infty} \left(4 - \frac{3}{x^2}\right) &= \lim_{x \to \infty} 4 - \lim_{x \to \infty} \frac{3}{x^2} \\
&= \lim_{x \to \infty} 4 - 3\left(\lim_{x \to \infty} \frac{1}{x^2}\right) \\
&= 4 - 3(0) \\
&= 4
\end{aligned}$$

So, the limit of

$$f(x) = 4 - \frac{3}{x^2}$$

as x approaches ∞ is 4.

Graphical Solution

Use a graphing utility to graph

$$y = 4 - \frac{3}{x^2}.$$

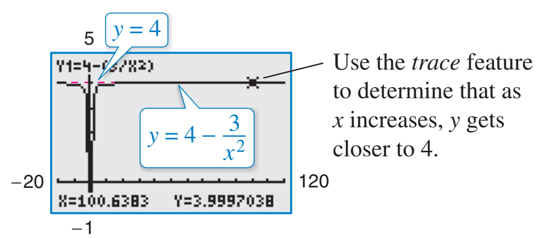


Figure 11.30

From Figure 11.30, you can estimate the limit to be 4. Note in the figure that the line $y = 4$ is a horizontal asymptote to the right.

CHECKPOINT Now try Exercise 13.

In Figure 11.30, it appears that the line

$y = 4$

is also a horizontal asymptote *to the left*. You can verify this by showing that

$$\lim_{x \to -\infty} \left(4 - \frac{3}{x^2}\right) = 4.$$

The graph of a rational function need not have a horizontal asymptote. When it does, however, its left and right asymptotes must be the same.

When evaluating limits at infinity for more complicated rational functions, divide the numerator and denominator by the *highest-powered term* in the denominator. This enables you to evaluate each limit using the limits at infinity at the top of this page.

Example 2 Comparing Limits at Infinity

Find the limit as x approaches ∞ for each function.

a. $f(x) = \dfrac{-2x + 3}{3x^2 + 1}$

b. $f(x) = \dfrac{-2x^2 + 3}{3x^2 + 1}$

c. $f(x) = \dfrac{-2x^3 + 3}{3x^2 + 1}$

Solution

In each case, begin by dividing both the numerator and denominator by

x^2

the highest-powered term in the denominator.

a. $$\lim_{x\to\infty} \frac{-2x + 3}{3x^2 + 1} = \lim_{x\to\infty} \frac{-\dfrac{2}{x} + \dfrac{3}{x^2}}{3 + \dfrac{1}{x^2}}$$

$$= \frac{-0 + 0}{3 + 0}$$

$$= 0$$

b. $$\lim_{x\to\infty} \frac{-2x^2 + 3}{3x^2 + 1} = \lim_{x\to\infty} \frac{-2 + \dfrac{3}{x^2}}{3 + \dfrac{1}{x^2}}$$

$$= \frac{-2 + 0}{3 + 0}$$

$$= -\frac{2}{3}$$

c. $$\lim_{x\to\infty} \frac{-2x^3 + 3}{3x^2 + 1} = \lim_{x\to\infty} \frac{-2x + \dfrac{3}{x^2}}{3 + \dfrac{1}{x^2}}$$

In this case, you can conclude that the limit does not exist because the numerator decreases without bound as the denominator approaches 3.

CHECKPOINT Now try Exercise 19.

In Example 2, observe that when the degree of the numerator is less than the degree of the denominator, as in part (a), the limit is 0. When the degrees of the numerator and denominator are equal, as in part (b), the limit is the ratio of the coefficients of the highest-powered terms. When the degree of the numerator is greater than the degree of the denominator, as in part (c), the limit does not exist.

This result seems reasonable when you realize that for large values of x, the highest-powered term of a polynomial is the most "influential" term. That is, a polynomial tends to behave as its highest-powered term behaves as x approaches positive or negative infinity.

Explore the Concept

Use a graphing utility to complete the table below to verify that

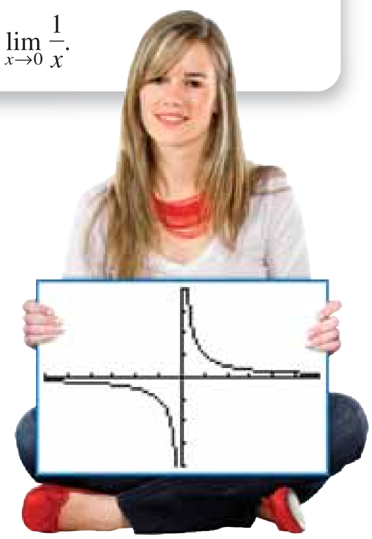
$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

x	10^0	10^1	10^2
$\dfrac{1}{x}$			

x	10^3	10^4	10^5
$\dfrac{1}{x}$			

Make a conjecture about

$$\lim_{x\to 0} \frac{1}{x}.$$



Activity

Have students use these observations from Example 2 to predict the following limits.

a. $\displaystyle\lim_{x\to\infty} \frac{5x(x - 3)}{2x}$

b. $\displaystyle\lim_{x\to\infty} \frac{4x^3 - 5x}{8x^4 + 3x^2 - 2}$

c. $\displaystyle\lim_{x\to\infty} \frac{-6x^2 + 1}{3x^2 + x - 2}$

Then ask several students to verify the predictions algebraically, several other students to verify the predictions numerically, and several more students to verify the predictions graphically. Lead a discussion comparing the results.

Limits at Infinity for Rational Functions

Consider the rational function

$$f(x) = \frac{N(x)}{D(x)}$$

where

$$N(x) = a_n x^n + \cdot \cdot \cdot + a_0 \quad \text{and} \quad D(x) = b_m x^m + \cdot \cdot \cdot + b_0.$$

The limit of $f(x)$ as x approaches positive or negative infinity is as follows.

$$\lim_{x \to \pm\infty} f(x) = \begin{cases} 0, & n < m \\ \dfrac{a_n}{b_m}, & n = m \end{cases}$$

If $n > m$, then the limit does not exist.

Example 3 Finding the Average Cost

You are manufacturing greeting cards that cost \$0.50 per card to produce. Your initial investment is \$5000, which implies that the total cost C of producing x cards is given by

$$C = 0.50x + 5000.$$

The average cost $\overline{C}$ per card is given by

$$\overline{C} = \frac{C}{x} = \frac{0.50x + 5000}{x}.$$

Find the average cost per card when (a) $x = 1000$, (b) $x = 10{,}000$, and (c) $x = 100{,}000$. (d) What is the limit of $\overline{C}$ as x approaches infinity?

Consider asking your students to identify the practical interpretation of the limit in part (d) of Example 3.

Solution

a. When $x = 1000$, the average cost per card is

$$\overline{C} = \frac{0.50(1000) + 5000}{1000} \qquad x = 1000$$

$$= \$5.50.$$

b. When $x = 10{,}000$, the average cost per card is

$$\overline{C} = \frac{0.50(10{,}000) + 5000}{10{,}000} \qquad x = 10{,}000$$

$$= \$1.00.$$

c. When $x = 100{,}000$, the average cost per card is

$$\overline{C} = \frac{0.50(100{,}000) + 5000}{100{,}000} \qquad x = 100{,}000$$

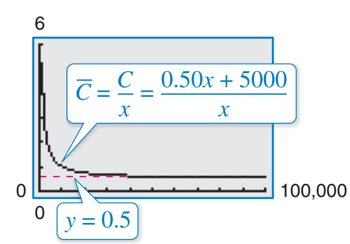
$$= \$0.55.$$

d. As x approaches infinity, the limit of $\overline{C}$ is

$$\lim_{x \to \infty} \frac{0.50x + 5000}{x} = \$0.50. \qquad x \to \infty$$

The graph of $\overline{C}$ is shown in Figure 11.31.

CHECKPOINT Now try Exercise 57.



As $x \to \infty$, the average cost per card approaches \$0.50.
Figure 11.31

Limits of Sequences

Limits of sequences have many of the same properties as limits of functions. For instance, consider the sequence whose nth term is $a_n = 1/2^n$.

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$$

As n increases without bound, the terms of this sequence get closer and closer to 0, and the sequence is said to **converge** to 0. Using limit notation, you can write

$$\lim_{n \to \infty} \frac{1}{2^n} = 0.$$

The following relationship shows how limits of functions of x can be used to evaluate the limit of a sequence.

Limit of a Sequence

Let L be a real number. Let f be a function of a real variable such that

$$\lim_{x \to \infty} f(x) = L.$$

If $\{a_n\}$ is a sequence such that

$$f(n) = a_n$$

for every positive integer n, then

$$\lim_{n \to \infty} a_n = L.$$

A sequence that does not converge is said to **diverge.** For instance, the sequence

$$1, -1, 1, -1, 1, \ldots$$

diverges because it does not approach a unique number.

Another sequence that diverges is $a_n = \frac{1}{n^{-1/4}}$. You might want your students to discuss why this is true.

Example 4 Finding the Limit of a Sequence

Find the limit of each sequence. (Assume n begins with 1.)

a. $a_n = \dfrac{2n + 1}{n + 4}$

b. $b_n = \dfrac{2n + 1}{n^2 + 4}$

c. $c_n = \dfrac{2n^2 + 1}{4n^2}$

Solution

a. $\displaystyle\lim_{n \to \infty} \frac{2n + 1}{n + 4} = 2$ $\qquad \frac{3}{5}, \frac{5}{6}, \frac{7}{7}, \frac{9}{8}, \frac{11}{9}, \frac{13}{10}, \ldots \to 2$

b. $\displaystyle\lim_{n \to \infty} \frac{2n + 1}{n^2 + 4} = 0$ $\qquad \frac{3}{5}, \frac{5}{8}, \frac{7}{13}, \frac{9}{20}, \frac{11}{29}, \frac{13}{40}, \ldots \to 0$

c. $\displaystyle\lim_{n \to \infty} \frac{2n^2 + 1}{4n^2} = \frac{1}{2}$ $\qquad \frac{3}{4}, \frac{9}{16}, \frac{19}{36}, \frac{33}{64}, \frac{51}{100}, \frac{73}{144}, \ldots \to \frac{1}{2}$

CHECKPOINT Now try Exercise 43.

Study Tip

You can use the definition of limits at infinity for rational functions on page 783 to verify the limits of the sequences in Example 4.

In the next section, you will encounter limits of sequences such as that shown in Example 5. A strategy for evaluating such limits is to begin by writing the nth term in standard rational function form. Then you can determine the limit by comparing the degrees of the numerator and denominator, as shown on page 783.

Example 5 Finding the Limit of a Sequence

Find the limit of the sequence whose nth term is

$$a_n = \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right].$$

Algebraic Solution

Begin by writing the nth term in standard rational function form—as the ratio of two polynomials.

$$a_n = \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] \qquad \text{Write original } n\text{th term.}$$

$$= \frac{8(n)(n+1)(2n+1)}{6n^3} \qquad \text{Multiply fractions.}$$

$$= \frac{8n^3 + 12n^2 + 4n}{3n^3} \qquad \text{Write in standard rational form.}$$

From this form, you can see that the degree of the numerator is equal to the degree of the denominator. So, the limit of the sequence is the ratio of the coefficients of the highest-powered terms.

$$\lim_{n\to\infty} \frac{8n^3 + 12n^2 + 4n}{3n^3} = \frac{8}{3}$$

CHECKPOINT Now try Exercise 53.

Numerical Solution

Enter the sequence

$$a_n = \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

into a graphing utility and create a table. (Be sure the graphing utility is set to *sequence* mode.)

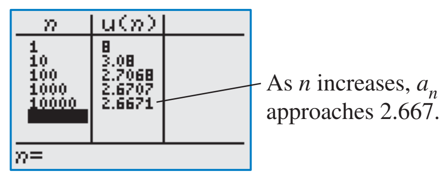


Figure 11.32

From Figure 11.32, you can estimate that as n approaches ∞, a_n gets closer and closer to $2.667 \approx \frac{8}{3}$.

The result of Example 5 is supported by Figure 11.33, which shows the graph of the sequence a_n and $y = 8/3$.

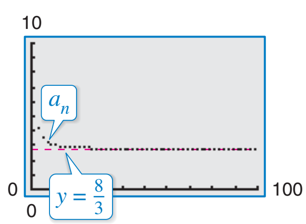


Figure 11.33

Explore the Concept

In Figure 11.32, the value of a_n approaches its limit of $\frac{8}{3}$ rather slowly. (The first term to be accurate to three decimal places is $a_{4801} \approx 2.667$.) Each of the following sequences converges to 0. Which converges the quickest? Which converges the slowest? Why? Write a short paragraph discussing your conclusions.

a. $a_n = \dfrac{1}{n}$ **b.** $b_n = \dfrac{1}{n^2}$ **c.** $c_n = \dfrac{1}{2^n}$ **d.** $d_n = \dfrac{1}{n!}$ **e.** $h_n = \dfrac{2^n}{n!}$

11.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

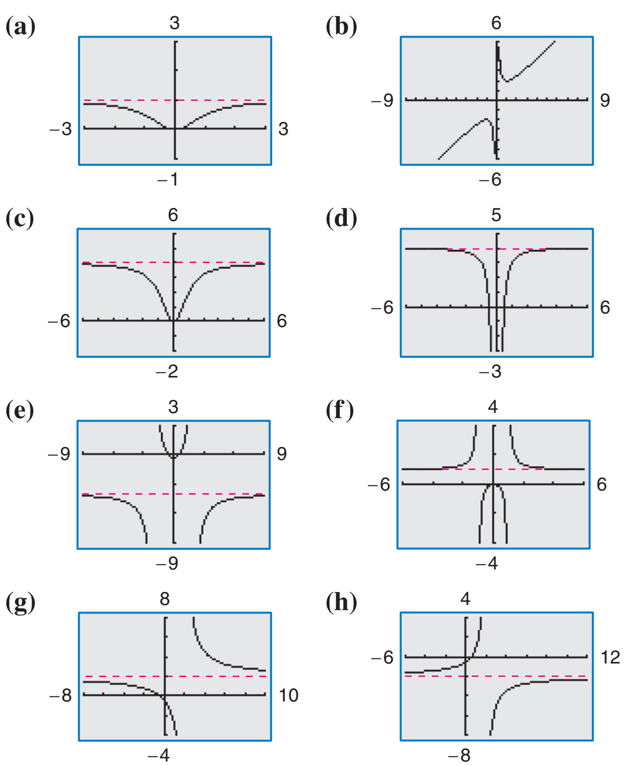
1. The line $y = 5$ is a horizontal asymptote to the right of the graph of a function f. What is the limit of f as x approaches infinity?
2. Given $\lim\limits_{x\to\infty} f(x) = 2$ for a rational function f, how does the degree of the numerator compare with the degree of the denominator?

In Exercises 3 and 4, fill in the blank.

3. A sequence that has a limit is said to ________ .
4. A sequence that does not have a limit is said to ________ .

Procedures and Problem Solving

Identifying the Graph of an Equation **In Exercises 5–12, match the function with its graph, using horizontal asymptotes as aids. [The graphs are labeled (a), (b), (c), (d), (e), (f), (g), and (h).]**



5. $f(x) = \dfrac{4x^2}{x^2 + 1}$
6. $f(x) = \dfrac{x^2}{x^2 + 1}$
7. $f(x) = 4 - \dfrac{1}{x^2}$
8. $f(x) = x + \dfrac{1}{x}$
9. $f(x) = \dfrac{x^2}{x^2 - 1}$
10. $f(x) = \dfrac{2x + 1}{x - 2}$
11. $f(x) = \dfrac{1 - 2x}{x - 2}$
12. $f(x) = \dfrac{1 - 4x^2}{x^2 - 4}$

Evaluating a Limit at Infinity **In Exercises 13–32, find the limit (if it exists). If the limit does not exist, explain why. Use a graphing utility to verify your result graphically.**

13. $\lim\limits_{x\to\infty} \dfrac{3}{x^2}$
14. $\lim\limits_{x\to\infty} \dfrac{5}{2x}$
15. $\lim\limits_{x\to\infty} \dfrac{3 + x}{3 - x}$
16. $\lim\limits_{x\to\infty} \dfrac{2 - 7x}{2 + 3x}$
17. $\lim\limits_{x\to-\infty} \dfrac{5x - 2}{6x + 1}$
18. $\lim\limits_{x\to-\infty} \dfrac{5 - 3x}{x + 4}$
19. $\lim\limits_{x\to-\infty} \dfrac{4x^2 - 3}{2 - x^2}$
20. $\lim\limits_{x\to-\infty} \dfrac{x^2 + 3}{5x^2 - 4}$
21. $\lim\limits_{t\to\infty} \dfrac{t^2}{t + 3}$
22. $\lim\limits_{y\to\infty} \dfrac{4y^4}{y^2 + 3}$
23. $\lim\limits_{t\to\infty} \dfrac{4t^2 + 3t - 1}{3t^2 + 2t - 5}$
24. $\lim\limits_{x\to\infty} \dfrac{5 - 6x - 3x^2}{2x^2 + x + 4}$
25. $\lim\limits_{y\to-\infty} \dfrac{3 + 8y - 4y^2}{3 - y - 2y^2}$
26. $\lim\limits_{t\to-\infty} \dfrac{t^2 + 9t - 10}{2 + 4t - 3t^2}$
27. $\lim\limits_{x\to-\infty} \dfrac{-(x^2 + 3)}{(2 - x)^2}$
28. $\lim\limits_{x\to\infty} \dfrac{2x^2 - 6}{(x - 1)^2}$
29. $\lim\limits_{x\to-\infty} \left[\dfrac{x}{(x + 1)^2} - 4\right]$
30. $\lim\limits_{x\to\infty} \left[7 + \dfrac{2x^2}{(x + 3)^2}\right]$
31. $\lim\limits_{t\to\infty} \left(\dfrac{1}{3t^2} - \dfrac{5t}{t + 2}\right)$
32. $\lim\limits_{x\to\infty} \left[\dfrac{x}{2x + 1} + \dfrac{3x^2}{(x - 3)^2}\right]$

Algebraic-Graphical-Numerical **In Exercises 33–38, (a) complete the table and numerically estimate the limit as x approaches infinity, (b) use a graphing utility to graph the function and estimate the limit graphically, and (c) find the limit algebraically.**

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$							

33. $f(x) = \dfrac{3x}{1 - x}$
34. $f(x) = \dfrac{x^2}{x^2 + 4}$

35. $f(x) = \dfrac{2x}{1 - x^2}$ **36.** $f(x) = \dfrac{2x + 1}{x^2 - 1}$

37. $f(x) = 1 - \dfrac{3}{x^2}$ **38.** $f(x) = 2 + \dfrac{1}{x}$

Estimating the Limit at Infinity **In Exercises 39–42, (a) complete the table and numerically estimate the limit as x approaches infinity and (b) use a graphing utility to graph the function and estimate the limit graphically.**

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$							

39. $f(x) = x - \sqrt{x^2 + 2}$ **40.** $f(x) = 3x - \sqrt{9x^2 + 1}$

41. $f(x) = 3\left(2x - \sqrt{4x^2 + x}\right)$

42. $f(x) = 4\left(4x - \sqrt{16x^2 - x}\right)$

Finding the Limit of a Sequence **In Exercises 43–52, write the first five terms of the sequence and find the limit of the sequence (if it exists). If the limit does not exist, explain why. Assume n begins with 1.**

43. $a_n = \dfrac{n + 1}{n^2 + 1}$ **44.** $a_n = \dfrac{n}{n^2 + 1}$

45. $a_n = \dfrac{n}{2n + 1}$ **46.** $a_n = \dfrac{4n - 1}{n + 3}$

47. $a_n = \dfrac{n^2}{3n + 2}$ **48.** $a_n = \dfrac{4n^2 + 1}{2n}$

49. $a_n = \dfrac{(n + 1)!}{n!}$ **50.** $a_n = \dfrac{(3n - 1)!}{(3n + 1)!}$

51. $a_n = \dfrac{(-1)^n}{n}$ **52.** $a_n = \dfrac{(-1)^{n+1}}{n^2}$

Finding the Limit of a Sequence **In Exercises 53–56, use a graphing utility to complete the table and estimate the limit of the sequence as n approaches infinity. Then find the limit algebraically.**

n	10^0	10^1	10^2	10^3
a_n				

53. $a_n = \dfrac{1}{n}\left(n + \dfrac{1}{n}\left[\dfrac{n(n + 1)}{2}\right]\right)$

54. $a_n = \dfrac{4}{n}\left(n + \dfrac{4}{n}\left[\dfrac{n(n + 1)}{2}\right]\right)$

55. $a_n = \dfrac{10}{n^3}\left[\dfrac{n(n + 1)(3n + 1)}{6}\right]$

56. $a_n = \dfrac{3n(n + 1)}{n^2} - \dfrac{4}{n^4}\left[\dfrac{n(n + 1)}{2}\right]^2$

57. Business The cost function for a certain model of a personal digital assistant (PDA) is given by $C = 13.50x + 45{,}750$, where C is the cost (in dollars) and x is the number of PDAs produced.

(a) Write a model for the average cost per unit produced.

(b) Find the average costs per unit when $x = 100$ and $x = 1000$.

(c) Determine the limit of the average cost function as x approaches infinity. Explain the meaning of the limit in the context of the problem.

58. Business The cost function for a new supermarket to recycle x tons of organic material is given by $C = 60x + 1650$, where C is the cost (in dollars).

(a) Write a model for the average cost per ton of organic material recycled.

(b) Find the average costs of recycling 100 tons and 1000 tons of organic material.

(c) Determine the limit of the average cost function as x approaches infinity. Explain the meaning of the limit in the context of the problem.

59. MODELING DATA

The table shows the numbers R (in thousands) of United States military reserve personnel in the years 2002 through 2008. (Source: U.S. Dept. of Defense)

Year	Reserves, R (in thousands)
2002	1222
2003	1189
2004	1167
2005	1136
2006	1120
2007	1110
2008	1100

A model for the data is given by

$$R(t) = \frac{61.018t^2 + 1260.64}{0.0578t^2 + 1}, \quad 2 \le t \le 8$$

where t represents the year, with $t = 2$ corresponding to 2002.

(a) Use a graphing utility to create a scatter plot of the data and graph the model in the same viewing window. How do they compare?

(b) Use the model to predict the numbers of reserves in 2009 and 2010.

(c) Find the limit of the model as $t \to \infty$ and interpret its meaning in the context of the situation.

(d) Is this a good model for predicting future numbers of reserves? Explain.

60. ***Why you should learn it*** *(p. 780)* The table shows the annual payrolls P (in millions of dollars) of the legislative branch of the United States government for the years 2001 through 2008. (Source: U.S. Office of Personnel Management)

Year	Payroll, P (in millions of dollars)
2001	1682
2002	1781
2003	1908
2004	1977
2005	2048
2006	2109
2007	2119
2008	2162

A model for the data is given by

$$P(t) = \frac{182.4312t^2 + 1634.39}{0.0808t^2 + 1}, \quad 1 \le t \le 8$$

where t represents the year, with $t = 1$ corresponding to 2001.

(a) Use a graphing utility to create a scatter plot of the data and graph the model in the same viewing window. How do they compare?

(b) Use the model to predict the payrolls in 2009 and 2010.

(c) Find the limit of the model as $t \to \infty$ and interpret its meaning in the context of the situation.

(d) Is this a good model for predicting the annual payrolls in future years? Explain.

Conclusions

True or False? **In Exercises 61–64, determine whether the statement is true or false. Justify your answer.**

61. Every rational function has a horizontal asymptote.

62. If a rational function f has a horizontal asymptote to the right, then the limit of $f(x)$ as x approaches $-\infty$ exists.

63. If a sequence converges, then it has a limit.

64. When the degrees of the numerator and denominator of a rational function are equal, the limit as x goes to infinity does not exist.

65. **Think About It** Find functions f and g such that both $f(x)$ and $g(x)$ increase without bound as x approaches ∞, but $\lim_{x\to\infty} [f(x) - g(x)] \neq \infty$.

66. **Think About It** Use a graphing utility to graph the function $f(x) = x/\sqrt{x^2 + 1}$. How many horizontal asymptotes does the function appear to have? What are the horizontal asymptotes?

Exploration **In Exercises 67–70, use a graphing utility to create a scatter plot of the terms of the sequence. Determine whether the sequence converges or diverges. If it converges, estimate its limit.**

67. $a_n = 4\left(\frac{2}{3}\right)^n$

68. $a_n = 3\left(\frac{3}{2}\right)^n$

69. $a_n = \dfrac{3[1 - (1.5)^n]}{1 - 1.5}$

70. $a_n = \dfrac{3[1 - (0.5)^n]}{1 - 0.5}$

71. **Error Analysis** Describe the error in finding the limit.

$$\cancel{\lim_{x\to\infty} \frac{1 - 2x - x^2}{4x^2 + 1} = 0}$$

72. **CAPSTONE** Let f be a rational function whose graph has the line $y = 3$ as a horizontal asymptote to the right.

(a) Find $\lim_{x\to\infty} f(x)$.

(b) Does the graph of f have a horizontal asymptote to the left? Explain your reasoning.

(c) Find $\lim_{x\to-\infty} f(x)$.

(d) Let $3x^3 - x + 4$ be the numerator of f. Which of the following expressions are possible denominators of f?

(i) $x^2 + 1$ (ii) $x^3 + 1$ (iii) $x^4 + 1$

Cumulative Mixed Review

Sketching Transformations **In Exercises 73 and 74, sketch the graphs of y and each transformation on the same rectangular coordinate system.**

73. $y = x^4$

(a) $f(x) = (x + 3)^4$ (b) $f(x) = x^4 - 1$

(c) $f(x) = -2 + x^4$ (d) $f(x) = \frac{1}{2}(x - 4)^4$

74. $y = x^3$

(a) $f(x) = (x + 2)^3$ (b) $f(x) = 3 + x^3$

(c) $f(x) = 2 - \frac{1}{4}x^3$ (d) $f(x) = 3(x + 1)^3$

Using Sigma Notation **In Exercises 75–78, find the sum.**

75. $\sum_{i=1}^{6} (2i + 3)$

76. $\sum_{i=0}^{4} 5i^2$

77. $\sum_{k=1}^{10} 15$

78. $\sum_{k=0}^{8} \dfrac{3}{k^2 + 1}$

11.5 The Area Problem

Limits of Summations

Earlier in the text, you used the concept of a limit to obtain a formula for the sum S of an infinite geometric series

$$S = a_1 + a_1 r + a_1 r^2 + \cdot \cdot \cdot = \sum_{i=1}^{\infty} a_1 r^{i-1} = \frac{a_1}{1-r}, \quad |r| < 1.$$

Using limit notation, this sum can be written as

$$\begin{aligned} S &= \lim_{n\to\infty} \sum_{i=1}^{n} a_1 r^{i-1} \\ &= \lim_{n\to\infty} \frac{a_1(1-r^n)}{1-r} \\ &= \frac{a_1}{1-r}. \qquad \lim_{n\to\infty} r^n = 0 \text{ for } 0 < |r| < 1 \end{aligned}$$

The following summation formulas and properties are used to evaluate finite and infinite summations.

Summation Formulas and Properties

1. $\displaystyle\sum_{i=1}^{n} c = cn$, c is a constant.
2. $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
3. $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
4. $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$
5. $\displaystyle\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$
6. $\displaystyle\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$, k is a constant.

Example 1 Evaluating a Summation

Evaluate the summation.

$$\sum_{i=1}^{200} i = 1 + 2 + 3 + 4 + \cdot \cdot \cdot + 200$$

Solution

Using Formula 2 with $n = 200$, you can write

$$\begin{aligned} \sum_{i=1}^{n} i &= \frac{n(n+1)}{2} \\ \sum_{i=1}^{200} i &= \frac{200(200+1)}{2} \\ &= \frac{40,200}{2} \\ &= 20,100. \end{aligned}$$

CHECKPOINT Now try Exercise 5.

What you should learn

- Find limits of summations.
- Use rectangles to approximate and limits of summations to find areas of plane regions.

Why you should learn it

Limits of summations are useful in determining areas of plane regions. For instance, in Exercise 46 on page 795, you are asked to find the limit of a summation to determine the area of a parcel of land bounded by a stream and two roads.

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Study Tip

Recall from Section 8.3 that the sum of a finite geometric sequence is given by

$$\sum_{i=1}^{n} a_1 r^{i-1} = a_1\left(\frac{1-r^n}{1-r}\right).$$

Furthermore, if $0 < |r| < 1$, then $r^n \to 0$ as $n \to \infty$.

Example 2 Evaluating a Summation

Evaluate the summation

$$\begin{aligned} S &= \sum_{i=1}^{n} \frac{i+2}{n^2} \\ &= \frac{3}{n^2} + \frac{4}{n^2} + \frac{5}{n^2} + \cdots + \frac{n+2}{n^2} \end{aligned}$$

for $n = 10, 100, 1000,$ and $10{,}000$.

Solution

Begin by applying summation formulas and properties to simplify S. In the second line of the solution, note that

$$\frac{1}{n^2}$$

can be factored out of the sum because n is considered to be constant. You could not factor i out of the summation because i is the (variable) index of summation.

$S = \sum_{i=1}^{n} \frac{i+2}{n^2}$	Write original form of summation.
$= \frac{1}{n^2} \sum_{i=1}^{n} (i+2)$	Factor constant $1/n^2$ out of sum.
$= \frac{1}{n^2} \left(\sum_{i=1}^{n} i + \sum_{i=1}^{n} 2 \right)$	Write as two sums.
$= \frac{1}{n^2} \left[\frac{n(n+1)}{2} + 2n \right]$	Apply Formulas 1 and 2.
$= \frac{1}{n^2} \left(\frac{n^2 + 5n}{2} \right)$	Add fractions.
$= \frac{n+5}{2n}$	Simplify.

Now you can evaluate the sum by substituting the appropriate values of n, as shown in the following table.

n	10	100	1000	10,000
$\sum_{i=1}^{n} \frac{i+2}{n^2} = \frac{n+5}{2n}$	0.75	0.525	0.5025	0.50025

✓CHECKPOINT Now try Exercise 15(a) and (b).

In Example 2, note that the sum appears to approach a limit as n increases. To find the limit of

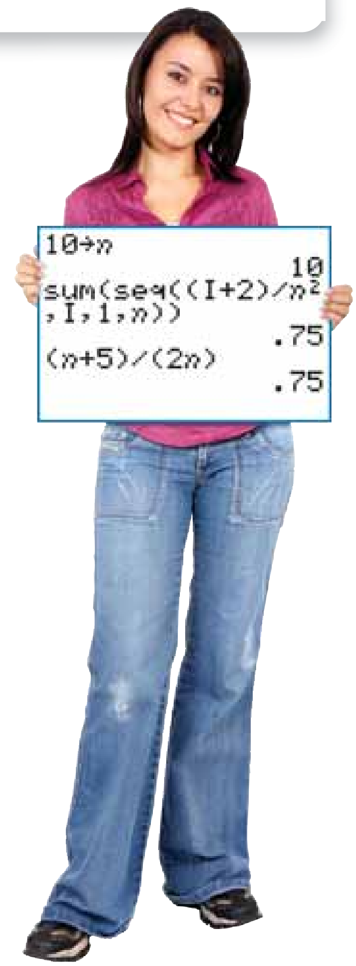
$$\frac{n+5}{2n}$$

as n approaches infinity, you can use the techniques from Section 11.4 to write

$$\lim_{n \to \infty} \frac{n+5}{2n} = \frac{1}{2}.$$

Technology Tip

Some graphing utilities have a *sum sequence* feature that is useful for computing summations. For instructions on how to use the *sum sequence* feature, see Appendix A; for specific keystrokes, go to this textbook's *Companion Website*.



Point out that finding the sums of progressively larger numbers of terms—i.e., larger values of n—will give better and better approximations of the limit of the summation at infinity. For instance, compute the sum in Example 2 for $n = 100{,}000$ and $n = 1{,}000{,}000$. What values are these sums approaching?

Be sure you notice the strategy used in Example 2. Rather than separately evaluating the sums

$$\sum_{i=1}^{10} \frac{i+2}{n^2}, \quad \sum_{i=1}^{100} \frac{i+2}{n^2}, \quad \sum_{i=1}^{1000} \frac{i+2}{n^2}, \quad \sum_{i=1}^{10,000} \frac{i+2}{n^2}$$

it was more efficient first to convert to rational form using the summation formulas and properties listed on page 789.

$$S = \underbrace{\sum_{i=1}^{n} \frac{i+2}{n^2}}_{\text{Summation form}} = \underbrace{\frac{n+5}{2n}}_{\text{Rational form}}$$

With this rational form, each sum can be evaluated by simply substituting appropriate values of n.

Example 3 Finding the Limit of a Summation

Find the limit of $S(n)$ as $n \to \infty$.

$$S(n) = \sum_{i=1}^{n} \left(1 + \frac{i}{n}\right)^2 \left(\frac{1}{n}\right)$$

Solution

Begin by rewriting the summation in rational form.

$$S(n) = \sum_{i=1}^{n} \left(1 + \frac{i}{n}\right)^2 \left(\frac{1}{n}\right) \qquad \text{Write original form of summation.}$$

$$= \sum_{i=1}^{n} \left(\frac{n^2 + 2ni + i^2}{n^2}\right)\left(\frac{1}{n}\right) \qquad \text{Square } (1 + i/n) \text{ and write as a single fraction.}$$

$$= \frac{1}{n^3} \sum_{i=1}^{n} (n^2 + 2ni + i^2) \qquad \text{Factor constant } 1/n^3 \text{ out of the sum.}$$

$$= \frac{1}{n^3}\left(\sum_{i=1}^{n} n^2 + \sum_{i=1}^{n} 2ni + \sum_{i=1}^{n} i^2\right) \qquad \text{Write as three sums.}$$

$$= \frac{1}{n^3}\left\{n^3 + 2n\left[\frac{n(n+1)}{2}\right] + \frac{n(n+1)(2n+1)}{6}\right\} \qquad \text{Use summation formulas.}$$

$$= \frac{14n^3 + 9n^2 + n}{6n^3} \qquad \text{Simplify.}$$

In this rational form, you can now find the limit as $n \to \infty$.

$$\lim_{n\to\infty} S(n) = \lim_{n\to\infty} \frac{14n^3 + 9n^2 + n}{6n^3} = \frac{14}{6} = \frac{7}{3}$$

CHECKPOINT Now try Exercise 15(c).

Study Tip

As you can see from Example 3, there is a lot of algebra involved in rewriting a summation in rational form. You may want to review simplifying rational expressions if you are having difficulty with this procedure.

The Area Problem

You now have the tools needed to solve the second basic problem of calculus: the area problem. The problem is to find the *area* of the region R bounded by the graph of a nonnegative, continuous function f, the x-axis, and the vertical lines $x = a$ and $x = b$, as shown in Figure 11.34.

When the region R is a square, a triangle, a trapezoid, or a semicircle, you can find its area by using a geometric formula. For more general regions, however, you must use a different approach—one that involves the limit of a summation. The basic strategy is to use a collection of rectangles of equal width that approximates the region R, as illustrated in Example 4.

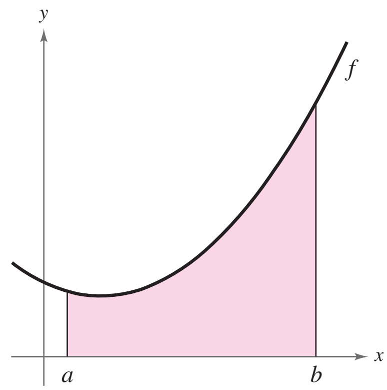


Figure 11.34

Example 4 Approximating the Area of a Region

Use the five rectangles in Figure 11.35 to approximate the area of the region bounded by the graph of

$$f(x) = 6 - x^2$$

the x-axis, and the lines $x = 0$ and $x = 2$.

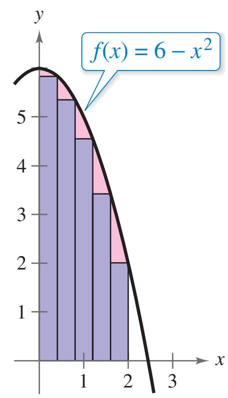


Figure 11.35

Solution

Because the length of the interval along the x-axis is 2 and there are five rectangles, the width of each rectangle is $\frac{2}{5}$. The height of each rectangle can be obtained by evaluating f at the right endpoint of each interval. The five intervals are as follows.

$$\left[0, \frac{2}{5}\right], \quad \left[\frac{2}{5}, \frac{4}{5}\right], \quad \left[\frac{4}{5}, \frac{6}{5}\right], \quad \left[\frac{6}{5}, \frac{8}{5}\right], \quad \left[\frac{8}{5}, \frac{10}{5}\right]$$

Notice that the right endpoint of each interval is $\frac{2}{5}i$ for $i = 1, 2, 3, 4,$ and 5. The sum of the areas of the five rectangles is

$$\begin{aligned}
\sum_{i=1}^{5} \underbrace{f\left(\frac{2i}{5}\right)}_{\text{Height}}\underbrace{\left(\frac{2}{5}\right)}_{\text{Width}} &= \sum_{i=1}^{5}\left[6 - \left(\frac{2i}{5}\right)^2\right]\left(\frac{2}{5}\right) \\
&= \frac{2}{5}\left(\sum_{i=1}^{5} 6 - \frac{4}{25}\sum_{i=1}^{5} i^2\right) \\
&= \frac{2}{5}\left(30 - \frac{44}{5}\right) \\
&= \frac{212}{25} \\
&= 8.48.
\end{aligned}$$

So, you can approximate the area of R as 8.48 square units.

CHECKPOINT Now try Exercise 21.

By increasing the number of rectangles used in Example 4, you can obtain closer and closer approximations of the area of the region. For instance, using 25 rectangles of width $\frac{2}{25}$ each, you can approximate the area to be

$$A \approx 9.17 \text{ square units.}$$

The following table shows even better approximations.

n	5	25	100	1000	5000
Approximate area	8.48	9.17	9.29	9.33	9.33

Consider leading a discussion on why increasing the number of rectangles used to approximate the area gives better and better estimates of the true area.

Based on the procedure illustrated in Example 4, the *exact* **area of a plane region** R is given by the limit of the sum of n rectangles as n approaches ∞.

Area of a Plane Region

Let f be continuous and nonnegative on the interval $[a, b]$. The **area** A of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{f\left(a + \frac{(b-a)i}{n}\right)}_{\text{Height}} \underbrace{\left(\frac{b-a}{n}\right)}_{\text{Width}}.$$

Example 5 Finding the Area of a Region

Find the area of the region bounded by the graph of

$$f(x) = x^2$$

and the x-axis between

$$x = 0 \quad \text{and} \quad x = 1$$

as shown in Figure 11.36.

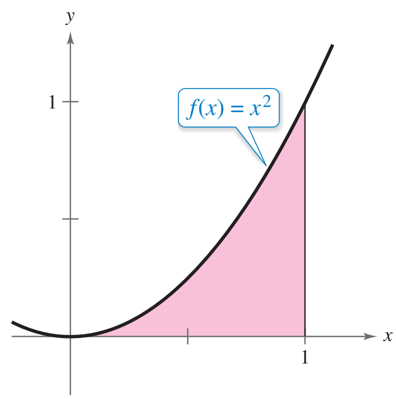


Figure 11.36

Solution

Begin by finding the dimensions of the rectangles.

$$\textit{Width:}\ \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

$$\textit{Height:}\ f\left(a + \frac{(b-a)i}{n}\right) = f\left(0 + \frac{(1-0)i}{n}\right) = f\left(\frac{i}{n}\right) = \frac{i^2}{n^2}$$

Next, approximate the area as the sum of the areas of n rectangles.

$$\begin{aligned} A &\approx \sum_{i=1}^{n} f\left(a + \frac{(b-a)i}{n}\right)\left(\frac{b-a}{n}\right) \\ &= \sum_{i=1}^{n} \left(\frac{i^2}{n^2}\right)\left(\frac{1}{n}\right) \\ &= \sum_{i=1}^{n} \frac{i^2}{n^3} \\ &= \frac{1}{n^3}\sum_{i=1}^{n} i^2 \\ &= \frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] \\ &= \frac{2n^3 + 3n^2 + n}{6n^3} \end{aligned}$$

Finally, find the exact area by taking the limit as n approaches ∞.

$$A = \lim_{n\to\infty} \frac{2n^3 + 3n^2 + n}{6n^3} = \frac{1}{3}$$

CHECKPOINT Now try Exercise 33.

11.5 Exercises

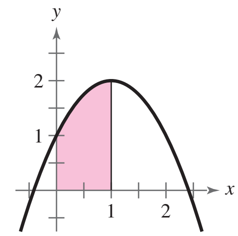
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blank.

1. $\sum_{i=1}^{n} i =$ ________

2. $\sum_{i=1}^{n} i^3 =$ ________



3. Can you obtain a better approximation of the area of the shaded region using 10 rectangles of equal width or 100 rectangles of equal width?

4. Does the limit of the sum of n rectangles as n approaches infinity represent the *exact* area of a plane region or an *approximation* of the area?

Procedures and Problem Solving

Evaluating a Summation **In Exercises 5–12, evaluate the sum using the summation formulas and properties.**

5. $\sum_{i=1}^{60} 7$

6. $\sum_{i=1}^{45} 3$

7. $\sum_{i=1}^{20} i^3$

8. $\sum_{i=1}^{30} i^2$

9. $\sum_{k=1}^{20} (k^3 + 2)$

10. $\sum_{k=1}^{50} (2k + 1)$

11. $\sum_{j=1}^{25} (j^2 + j)$

12. $\sum_{j=1}^{10} (j^3 - 3j^2)$

Finding the Limit of a Summation **In Exercises 13–18, (a) rewrite the sum as a rational function $S(n)$. (b) Use $S(n)$ to complete the table. (c) Find $\lim_{n \to \infty} S(n)$.**

n	10^0	10^1	10^2	10^3	10^4
$S(n)$					

13. $\sum_{i=1}^{n} \frac{i^3}{n^4}$

14. $\sum_{i=1}^{n} \frac{i}{n^2}$

15. $\sum_{i=1}^{n} \frac{3}{n^3}(1 + i^2)$

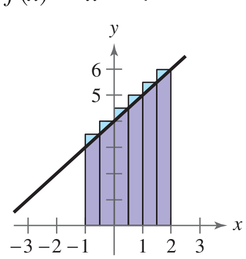
16. $\sum_{i=1}^{n} \frac{2i + 3}{n^2}$

17. $\sum_{i=1}^{n} \left(\frac{i^2}{n^3} + \frac{2}{n}\right)\left(\frac{1}{n}\right)$

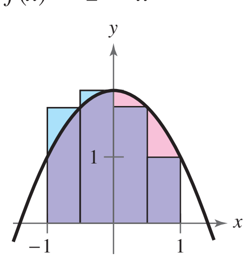
18. $\sum_{i=1}^{n} \left[3 - 2\left(\frac{i}{n}\right)\right]\left(\frac{1}{n}\right)$

Approximating the Area of a Region **In Exercises 19–22, approximate the area of the region using the indicated number of rectangles of equal width.**

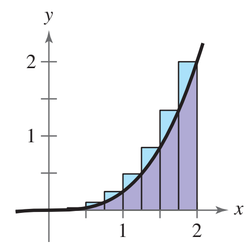
19. $f(x) = x + 4$



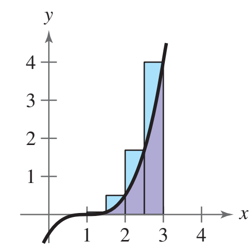
20. $f(x) = 2 - x^2$



21. $f(x) = \frac{1}{4}x^3$
(8 rectangles)



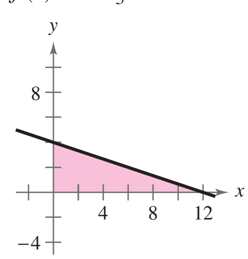
22. $f(x) = \frac{1}{2}(x - 1)^3$
(4 rectangles)



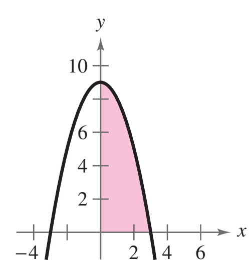
Approximating the Area of a Region **In Exercises 23–26, complete the table to show the approximate area of the region using the indicated numbers n of rectangles of equal width.**

n	4	8	20	50
Approximate area				

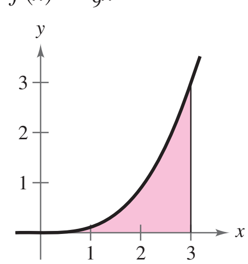
23. $f(x) = -\frac{1}{3}x + 4$



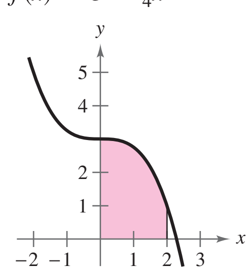
24. $f(x) = 9 - x^2$



25. $f(x) = \frac{1}{9}x^3$



26. $f(x) = 3 - \frac{1}{4}x^3$



The Area of a Region **In Exercises 27–32, use the given expression for the sum of the areas of n rectangles. For each finite value of n in the table, approximate the area of the region bounded by the graph of f and the x-axis over the specified interval. Then find the exact area as $n \to \infty$.**

n	4	8	20	50	100	∞
Area						

	Function	*Interval*	*Sum of areas of n rectangles*
27.	$f(x) = 2x + 5$	$[0, 4]$	$\dfrac{36n^2 + 16n}{n^2}$
28.	$f(x) = 3x + 1$	$[0, 4]$	$\dfrac{28n^2 + 24n}{n^2}$
29.	$f(x) = 9 - x^2$	$[0, 2]$	$\dfrac{46n^3 - 12n^2 - 4n}{3n^3}$
30.	$f(x) = x^2 + 1$	$[4, 6]$	$\dfrac{158n^3 + 60n^2 + 4n}{3n^3}$
31.	$f(x) = \frac{1}{2}x + 4$	$[-1, 3]$	$\dfrac{18n^2 + 4n}{n^2}$
32.	$f(x) = \frac{1}{2}x + 1$	$[-2, 2]$	$\dfrac{4n^2 + 4n}{n^2}$

Finding the Area of a Region **In Exercises 33–44, use the limit process to find the area of the region bounded by the graph of the function and the x-axis over the specified interval.**

	Function	*Interval*
33.	$f(x) = 4x + 1$	$[0, 1]$
34.	$f(x) = 3x + 2$	$[0, 2]$
35.	$f(x) = -x + 4$	$[0, 3]$
36.	$f(x) = 3x - 6$	$[2, 5]$
37.	$f(x) = 16 - x^2$	$[0, 4]$
38.	$f(x) = x^2 + 2$	$[0, 1]$
39.	$g(x) = 1 - x^3$	$[0, 1]$
40.	$g(x) = 64 - x^3$	$[0, 3]$
41.	$g(x) = 2x - x^3$	$[0, 1]$
42.	$g(x) = 4x - x^3$	$[0, 2]$
43.	$f(x) = x^2 + 4x$	$[0, 6]$
44.	$f(x) = x^2 - x^3$	$[0, 1]$

45. **Geometry** The boundaries of a parcel of land are two edges modeled by the coordinate axes and a stream modeled by the equation

$$y = (-3.0 \times 10^{-6})x^3 + 0.002x^2 - 1.05x + 400.$$

Use a graphing utility to graph the equation. Find the area of the property. (All distances are measured in feet.)

46. ***Why you should learn it*** *(p. 789)* The table shows the measurements (in feet) of a lot bounded by a stream and two straight roads that meet at right angles (see figure).

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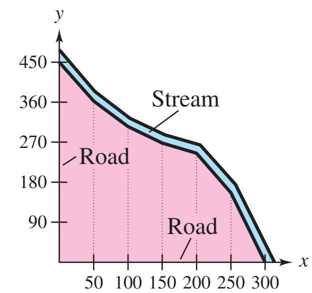
x	0	50	100	150
y	450	362	305	268

x	200	250	300
y	245	156	0



(a) Use the *regression* feature of a graphing utility to find a model of the form $y = ax^3 + bx^2 + cx + d$.

(b) Use the graphing utility to plot the data and graph the model in the same viewing window.

(c) Use the model in part (a) to estimate the area of the lot.

Conclusions

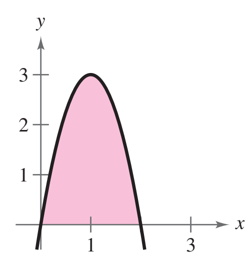
True or False? **In Exercises 47 and 48, determine whether the statement is true or false. Justify your answer.**

47. The sum of the first n positive integers is $n(n + 1)/2$.

48. The exact area of a region is given by the limit of the sum of n rectangles as n approaches 0.

49. **Think About It** Determine which value best approximates the area of the region shown in the graph. (Make your selection on the basis of the sketch of the region and *not* by performing any calculations.)

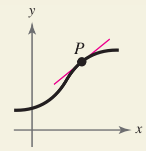
(a) -2
(b) 1
(c) 4
(d) 6
(e) 9



50. **CAPSTONE** Describe the process of finding the areas of a region bounded by the graph of a nonnegative, continuous function f, the x-axis, and the vertical lines $x = a$ and $x = b$.

11 Chapter Summary

<table>
<tr><th></th><th>What did you learn?</th><th>Explanation and Examples</th><th>Review Exercises</th></tr>
<tr><td rowspan="3">11.1</td><td>Understand the limit concept (p. 750) and use the definition of a limit to estimate limits (p. 751).</td><td>If $f(x)$ becomes arbitrarily close to a unique number L as x approaches c from either side, then the limit of $f(x)$ as x approaches c is L. This is written as $\lim_{x \to c} f(x) = L$.</td><td>1–4</td></tr>
<tr><td>Determine whether limits of functions exist (p. 753).</td><td>Conditions Under Which Limits Do Not Exist
The limit of $f(x)$ as $x \to c$ does not exist under any of the following conditions.
1. $f(x)$ approaches a different number from the right side of c than it approaches from the left side of c.
2. $f(x)$ increases or decreases without bound as x approaches c.
3. $f(x)$ oscillates between two fixed values as x approaches c.</td><td>5–8</td></tr>
<tr><td>Use properties of limits and direct substitution to evaluate limits (p. 755).</td><td>Let b and c be real numbers and let n be a positive integer.
1. $\lim_{x \to c} b = b$
2. $\lim_{x \to c} x = c$
3. $\lim_{x \to c} x^n = c^n$
4. $\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$, for n even and $c > 0$
Properties of Limits
Let b and c be real numbers, let n be a positive integer, and let f and g be functions where
$\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = K$.
1. $\lim_{x \to c} [bf(x)] = bL$ 2. $\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$
3. $\lim_{x \to c} [f(x)g(x)] = LK$ 4. $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{K}, \quad K \neq 0$
5. $\lim_{x \to c} [f(x)]^n = L^n$</td><td>9–22</td></tr>
<tr><td rowspan="5">11.2</td><td>Use the dividing out technique to evaluate limits of functions (p. 760).</td><td>When evaluating a limit of a rational function by direct substitution, you may encounter the indeterminate form $0/0$. In this case, factor and divide out any common factors, then try direct substitution again. (See Examples 1 and 2.)</td><td>23–30</td></tr>
<tr><td>Use the rationalizing technique to evaluate limits of functions (p. 762).</td><td>The rationalizing technique involves rationalizing the numerator of the function when finding a limit. (See Example 3.)</td><td>31, 32</td></tr>
<tr><td>Use technology to approximate limits of functions graphically and numerically (p. 763).</td><td>The table feature or zoom and trace features of a graphing utility can be used to approximate limits. (See Examples 4 and 5.)</td><td>33–40</td></tr>
<tr><td>Evaluate one-sided limits of functions (p. 764).</td><td>Limit from left: $\lim_{x \to c^-} f(x) = L_1$ or $f(x) \to L_1$ as $x \to c^-$
Limit from right: $\lim_{x \to c^+} f(x) = L_2$ or $f(x) \to L_2$ as $x \to c^+$</td><td>41–48</td></tr>
<tr><td>Evaluate limits of difference quotients from calculus (p. 766).</td><td>For any x-value, the limit of a difference quotient is an expression of the form $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.</td><td>49, 50</td></tr>
</table>

	What did you learn?	Explanation and Examples	Review Exercises
11.3	**Understand the tangent line problem *(p. 770)* and use a tangent line to approximate the slope of a graph at a point *(p. 771)*.**	The tangent line to the graph of a function f at a point $P(x_1, y_1)$ is the line that best approximates the slope of the graph at the point. 	51–58
	Use the limit definition of slope to find exact slopes of graphs *(p. 772)*.	**Definition of the Slope of a Graph** The slope m of the graph of f at the point $(x, f(x))$ is equal to the slope of its tangent line at $(x, f(x))$, and is given by $m = \lim_{h \to 0} m_{\text{sec}} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ provided this limit exists.	59–62
	Find derivatives of functions and use derivatives to find slopes of graphs *(p. 775)*.	The derivative of f at x is given by $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ provided this limit exists. The derivative $f'(x)$ is a formula for the slope of the tangent line to the graph of f at the point $(x, f(x))$.	63–78
11.4	**Evaluate limits of functions at infinity *(p. 780)*.**	If f is a function and L_1 and L_2 are real numbers, then the statements $\lim_{x \to -\infty} f(x) = L_1$ and $\lim_{x \to \infty} f(x) = L_2$ denote the limits at infinity.	79–86
	Find limits of sequences *(p. 784)*.	**Limit of a Sequence** Let L be a real number. Let f be a function of a real variable such that $\lim_{x \to \infty} f(x) = L$. If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer n, then $\lim_{n \to \infty} a_n = L$.	87–92
11.5	**Find limits of summations *(p. 789)*.**	**Summation Formulas and Properties** **1.** $\sum_{i=1}^{n} c = cn$, c is a constant. **2.** $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ **3.** $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ **4.** $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ **5.** $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$ **6.** $\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$, k is a constant.	93, 94
	Use rectangles to approximate and limits of summations to find areas of plane regions *(p. 792)*.	A collection of rectangles of equal width can be used to approximate the area of a region. Increasing the number of rectangles gives a closer approximation. (See Example 4.) **Area of a Plane Region** Let f be continuous and nonnegative on $[a, b]$. The area A of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by $A = \lim_{n \to \infty} \sum_{i=1}^{n} f\left(a + \frac{(b-a)i}{n}\right)\left(\frac{b-a}{n}\right).$	95–105

11 Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

11.1

Estimating a Limit Numerically **In Exercises 1–4, complete the table and use the result to estimate the limit numerically. Determine whether the limit can be reached.**

1. $\lim_{x\to 3}(6x - 1)$

x	2.9	2.99	2.999	3	3.001	3.01	3.1
$f(x)$				?			

2. $\lim_{x\to 2} \dfrac{x-2}{3x^2 - 4x - 4}$

x	1.9	1.99	1.999	2	2.001	2.01	2.1
$f(x)$				?			

3. $\lim_{x\to 0} \dfrac{1 - e^{-x}}{x}$

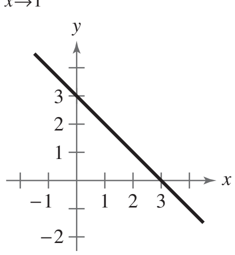
x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$				?			

4. $\lim_{x\to 0} \dfrac{\ln(1 - x)}{x}$

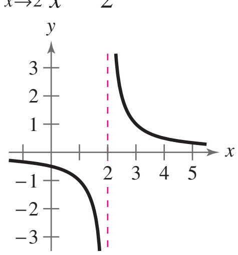
x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$				?			

Using a Graph to Find a Limit **In Exercises 5–8, use the graph to find the limit (if it exists). If the limit does not exist, explain why.**

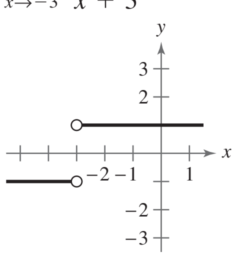
5. $\lim_{x\to 1}(3 - x)$



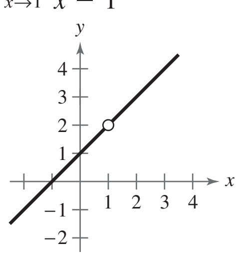
6. $\lim_{x\to 2} \dfrac{1}{x-2}$



7. $\lim_{x\to -3} \dfrac{|x+3|}{x+3}$



8. $\lim_{x\to 1} \dfrac{x^2 - 1}{x - 1}$



Evaluating Limits **In Exercises 9 and 10, use the given information to evaluate each limit.**

9. $\lim_{x\to c} f(x) = 2,\ \lim_{x\to c} g(x) = 5$

(a) $\lim_{x\to c}[f(x)]^3$ (b) $\lim_{x\to c}[3f(x) - g(x)]$

(c) $\lim_{x\to c}[f(x)g(x)]$ (d) $\lim_{x\to c} \dfrac{f(x)}{g(x)}$

10. $\lim_{x\to c} f(x) = 8,\ \lim_{x\to c} g(x) = 3$

(a) $\lim_{x\to c} \sqrt[3]{f(x)}$ (b) $\lim_{x\to c} \dfrac{f(x)}{18}$

(c) $\lim_{x\to c}[f(x)g(x)]$ (d) $\lim_{x\to c}[f(x) - 2g(x)]$

Evaluating Limits by Direct Substitution **In Exercises 11–22, find the limit by direct substitution.**

11. $\lim_{x\to 4}\left(\frac{1}{2}x + 3\right)$ **12.** $\lim_{x\to 3}(5x - 4)$

13. $\lim_{t\to 3} \dfrac{t^2 + 1}{t}$ **14.** $\lim_{x\to 2} \dfrac{3x + 5}{5x - 3}$

15. $\lim_{x\to -2} \sqrt[3]{4x}$ **16.** $\lim_{x\to -1} \sqrt{5 - x}$

17. $\lim_{x\to \pi} \sin 3x$ **18.** $\lim_{x\to 0} \tan x$

19. $\lim_{x\to -1} e^{-x}$ **20.** $\lim_{x\to 3} 2 \ln x$

21. $\lim_{x\to -1/2} \arcsin x$ **22.** $\lim_{x\to 0} \arctan x$

11.2

Finding Limits **In Exercises 23–32, find the limit (if it exists). Use a graphing utility to confirm your result graphically.**

23. $\lim_{t\to -2} \dfrac{t+2}{t^2 - 4}$ **24.** $\lim_{t\to 3} \dfrac{t^2 - 9}{t - 3}$

25. $\lim_{x\to 5} \dfrac{x - 5}{x^2 + 5x - 50}$ **26.** $\lim_{x\to -1} \dfrac{x + 1}{x^2 - 5x - 6}$

27. $\lim_{x\to 1} \dfrac{x^2 + 7x - 8}{x^2 - 3x + 2}$ **28.** $\lim_{x\to -2} \dfrac{x^2 + 8x + 12}{x^2 - 3x - 10}$

29. $\lim_{x\to -1} \dfrac{\dfrac{1}{x+2} - 1}{x + 1}$ **30.** $\lim_{x\to 0} \dfrac{\dfrac{1}{x+1} - 1}{x}$

31. $\lim_{u\to 0} \dfrac{\sqrt{4 + u} - 2}{u}$ **32.** $\lim_{v\to 0} \dfrac{\sqrt{v + 9} - 3}{v}$

Approximating a Limit **In Exercises 33–40, (a) graphically approximate the limit (if it exists) by using a graphing utility to graph the function and (b) numerically approximate the limit (if it exists) by using the *table* feature of the graphing utility to create a table.**

33. $\lim_{x\to 3} \dfrac{x - 3}{x^2 - 9}$ **34.** $\lim_{x\to 4} \dfrac{4 - x}{16 - x^2}$

35. $\lim_{x\to 0} e^{-2/x}$

36. $\lim_{x\to 0} e^{-4/x^2}$

37. $\lim_{x\to 0} \dfrac{\sin 4x}{2x}$

38. $\lim_{x\to 0} \dfrac{\tan 2x}{x}$

39. $\lim_{x\to 1^+} \dfrac{\sqrt{2x+1}-\sqrt{3}}{x-1}$

40. $\lim_{x\to 1^+} \dfrac{1-\sqrt{x}}{x-1}$

Evaluating One-Sided Limits **In Exercises 41–48, graph the function. Determine the limit (if it exists) by evaluating the corresponding one-sided limits.**

41. $\lim_{x\to 3} \dfrac{|x-3|}{x-3}$

42. $\lim_{x\to 8} \dfrac{|8-x|}{8-x}$

43. $\lim_{x\to 2} \dfrac{2}{x^2-4}$

44. $\lim_{x\to -3} \dfrac{1}{x^2+9}$

45. $\lim_{x\to 5} \dfrac{|x-5|}{x-5}$

46. $\lim_{x\to -2} \dfrac{|x+2|}{x+2}$

47. $\lim_{x\to 2} f(x)$ where $f(x) = \begin{cases} 5-x, & x \le 2 \\ x^2-3, & x > 2 \end{cases}$

48. $\lim_{x\to 0} f(x)$ where $f(x) = \begin{cases} x-6, & x \ge 0 \\ x^2-4, & x < 0 \end{cases}$

Evaluating a Limit from Calculus **In Exercises 49 and 50, find $\lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$.**

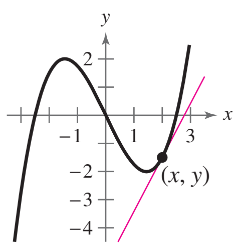
49. $f(x) = 3x - x^2$

50. $f(x) = x^2 - 5x - 2$

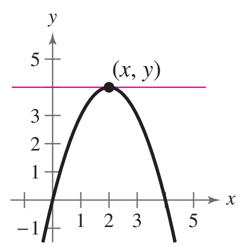
11.3

Approximating the Slope of a Graph **In Exercises 51 and 52, use the figure to approximate the slope of the curve at the point (x, y).**

51.



52.



Approximating the Slope of a Tangent Line **In Exercises 53–58, use a graphing utility to graph the function and the tangent line at the point $(2, f(2))$. Use the graph to approximate the slope of the tangent line.**

53. $f(x) = x^2 - 2x$

54. $f(x) = 6 - x^2$

55. $f(x) = \sqrt{x+2}$

56. $f(x) = \sqrt{x^2+5}$

57. $f(x) = \dfrac{6}{x-4}$

58. $f(x) = \dfrac{1}{3-x}$

Finding a Formula for the Slope of a Graph **In Exercises 59–62, find a formula for the slope of the graph of f at the point $(x, f(x))$. Then use it to find the slopes at the two specified points.**

59. $f(x) = x^2 - 4x$
   (a) $(0, 0)$
   (b) $(5, 5)$

60. $f(x) = \frac{1}{4}x^4$
   (a) $(-2, 4)$
   (b) $\left(1, \frac{1}{4}\right)$

61. $f(x) = \dfrac{4}{x-6}$
   (a) $(7, 4)$
   (b) $(8, 2)$

62. $f(x) = \sqrt{x}$
   (a) $(1, 1)$
   (b) $(4, 2)$

Finding a Derivative **In Exercises 63–74, find the derivative of the function.**

63. $f(x) = 5$

64. $g(x) = -3$

65. $h(x) = 5 - \frac{1}{2}x$

66. $f(x) = 3x$

67. $g(x) = 2x^2 - 1$

68. $f(x) = -x^3 + 4x$

69. $f(t) = \sqrt{t+5}$

70. $g(t) = \sqrt{t-3}$

71. $g(s) = \dfrac{4}{s+5}$

72. $g(t) = \dfrac{6}{5-t}$

73. $g(x) = \dfrac{1}{\sqrt{x+4}}$

74. $f(x) = \dfrac{1}{\sqrt{12-x}}$

Using the Derivative **In Exercises 75–78, (a) find the slope of the graph of f at the given point, (b) find an equation of the tangent line to the graph at the point, and (c) graph the function and the tangent line.**

75. $f(x) = 2x^2 - 1, (0, -1)$

76. $f(x) = x^2 + 10, (2, 14)$

77. $f(x) = x^3 + 1, (-1, 0)$

78. $f(x) = x^3 - x, (2, 6)$

11.4

Evaluating a Limit at Infinity **In Exercises 79–86, find the limit (if it exists). If the limit does not exist, explain why. Use a graphing utility to verify your result graphically.**

79. $\lim_{x\to\infty} \dfrac{4x}{2x-3}$

80. $\lim_{x\to\infty} \dfrac{7x}{14x+2}$

81. $\lim_{x\to -\infty} \dfrac{2x}{x^2-25}$

82. $\lim_{x\to -\infty} \dfrac{3x}{(1-x)^3}$

83. $\lim_{x\to\infty} \dfrac{x^2}{2x+3}$

84. $\lim_{y\to\infty} \dfrac{3y^4}{y^2+1}$

85. $\lim_{x\to\infty} \left[\dfrac{x}{(x-2)^2} + 3\right]$

86. $\lim_{x\to\infty} \left[2 - \dfrac{2x^2}{(x+1)^2}\right]$

Finding the Limit of a Sequence **In Exercises 87–92, write the first five terms of the sequence and find the limit of the sequence (if it exists). If the limit does not exist, explain why. Assume n begins with 1.**

87. $a_n = \dfrac{2n - 3}{5n + 4}$ **88.** $a_n = \dfrac{2n}{n^2 + 1}$

89. $a_n = \dfrac{(-1)^n}{n^3}$ **90.** $a_n = \dfrac{(-1)^{n+1}}{n}$

91. $a_n = \dfrac{1}{2n^2}[3 - 2n(n + 1)]$

92. $a_n = \left(\dfrac{2}{n}\right)\left\{n + \dfrac{2}{n}\left[\dfrac{n(n - 1)}{2} - n\right]\right\}$

11.5

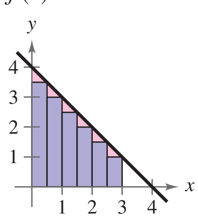
Finding the Limit of a Summation **In Exercises 93 and 94, (a) use the summation formulas and properties to rewrite the sum as a rational function $S(n)$. (b) Use $S(n)$ to complete the table. (c) Find $\lim_{n \to \infty} S(n)$.**

n	10^0	10^1	10^2	10^3	10^4
$S(n)$					

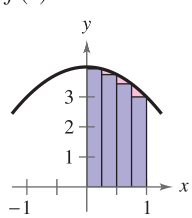
93. $\displaystyle\sum_{i=1}^{n} \left(\frac{4i^2}{n^2} - \frac{i}{n}\right)\left(\frac{1}{n}\right)$ **94.** $\displaystyle\sum_{i=1}^{n} \left[4 - \left(\frac{3i}{n}\right)^2\right]\left(\frac{3i}{n^2}\right)$

Approximating the Area of a Region **In Exercises 95 and 96, approximate the area of the region using the indicated number of rectangles of equal width.**

95. $f(x) = 4 - x$



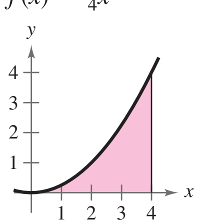
96. $f(x) = 4 - x^2$



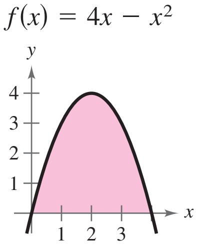
Approximating the Area of a Region **In Exercises 97 and 98, complete the table to show the approximate area of the region using the indicated numbers n of rectangles of equal width.**

n	4	8	20	50
Approximate area				

97. $f(x) = \frac{1}{4}x^2$



98. $f(x) = 4x - x^2$



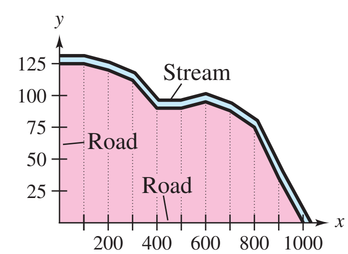
Finding the Area of a Region **In Exercises 99–104, use the limit process to find the area of the region bounded by the graph of the function and the x-axis over the specified interval.**

	Function	*Interval*
99.	$f(x) = 10 - x$	$[0, 10]$
100.	$f(x) = 2x - 6$	$[3, 6]$
101.	$f(x) = x^2 + 4$	$[0, 3]$
102.	$f(x) = 6(x - x^2)$	$[0, 1]$
103.	$f(x) = x^3 + 1$	$[0, 4]$
104.	$f(x) = 8 - x^3$	$[0, 2]$

105. Civil Engineering The table shows the measurements (in feet) of a lot bounded by a stream and two straight roads that meet at right angles (see figure).

x	0	100	200	300	400	500
y	125	125	120	112	90	90

x	600	700	800	900	1000
y	95	88	75	35	0



(a) Use the *regression* feature of a graphing utility to find a model of the form

$y = ax^3 + bx^2 + cx + d.$

(b) Use the graphing utility to plot the data and graph the model in the same viewing window.

(c) Use the model in part (a) to estimate the area of the lot.

Conclusions

True or False? **In Exercises 106 and 107, determine whether the statement is true or false. Justify your answer.**

106. If the degree of the numerator $N(x)$ of a rational function $f(x) = N(x)/D(x)$ is greater than the degree of its denominator $D(x)$, then the limit of the rational function as x approaches ∞ is 0.

107. The expression $f'(z)$ gives the slope of the tangent line to the graph of f at the point $(z, f(z))$.

108. Writing Write a short paragraph explaining several reasons why the limit of a function may not exist.

11 Chapter Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Take this test as you would take a test in class. After you are finished, check your work against the answers in the back of the book.

In Exercises 1–3, use a graphing utility to graph the function and approximate the limit (if it exists). Then find the limit (if it exists) algebraically by using appropriate techniques.

1. $\lim\limits_{x \to -2} \dfrac{x^2 - 1}{2x}$ **2.** $\lim\limits_{x \to 1} \dfrac{-x^2 + 5x - 3}{1 - x}$ **3.** $\lim\limits_{x \to 5} \dfrac{\sqrt{x} - 2}{x - 5}$

In Exercises 4 and 5, use a graphing utility to graph the function and approximate the limit. Write an approximation that is accurate to four decimal places. Then create a table to verify your limit numerically.

4. $\lim\limits_{x \to 0} \dfrac{\sin 3x}{x}$ **5.** $\lim\limits_{x \to 0} \dfrac{e^{2x} - 1}{x}$

6. Find a formula for the slope of the graph of f at the point $(x, f(x))$. Then use it to find the slope at the specified point.

(a) $f(x) = 3x^2 - 5x - 2,\ (2, 0)$ (b) $f(x) = 2x^3 + 6x,\ (-1, -8)$

In Exercises 7–9, find the derivative of the function.

7. $f(x) = 3 - \dfrac{2}{5}x$ **8.** $f(x) = 2x^2 + 4x - 1$ **9.** $f(x) = \dfrac{1}{x + 1}$

In Exercises 10–12, find the limit (if it exists). If the limit does not exist, explain why. Use a graphing utility to verify your result graphically.

10. $\lim\limits_{x \to \infty} \dfrac{6}{5x - 1}$ **11.** $\lim\limits_{x \to \infty} \dfrac{1 - 3x^2}{x^2 - 5}$ **12.** $\lim\limits_{x \to -\infty} \dfrac{x^2}{3x + 2}$

In Exercises 13 and 14, write the first five terms of the sequence and find the limit of the sequence (if it exists). If the limit does not exist, explain why. Assume *n* begins with 1.

13. $a_n = \dfrac{n^2 + 3n - 4}{2n^2 + n - 2}$ **14.** $a_n = \dfrac{1 + (-1)^n}{n}$

15. Approximate the area of the region bounded by the graph of $f(x) = 8 - 2x^2$ shown at the right using the indicated number of rectangles of equal width.

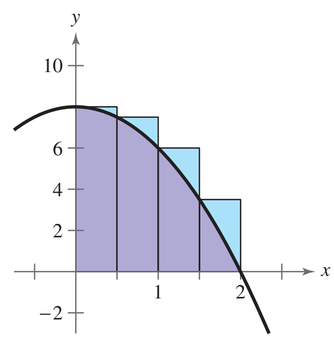


Figure for 15

In Exercises 16 and 17, use the limit process to find the area of the region bounded by the graph of the function and the *x*-axis over the specified interval.

16. $f(x) = x + 2$; interval: $[-2, 2]$

17. $f(x) = 7 - x^2$; interval: $[0, 2]$

18. The table shows the height of a space shuttle during its first 5 seconds of motion.

(a) Use the *regression* feature of a graphing utility to find a quadratic model $y = ax^2 + bx + c$ for the data.

(b) The value of the derivative of the model is the rate of change of height with respect to time, or the velocity, at that instant. Find the velocity of the shuttle after 5 seconds.

Time (seconds), x	Height (feet), y
0	0
1	1
2	23
3	60
4	115
5	188

Table for 18

10–11 Cumulative Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Take this test to review the material in Chapters 10 and 11. After you are finished, check your work against the answers in the back of the book.

In Exercises 1 and 2, find the coordinates of the point.

1. The point is located six units behind the yz-plane, one unit to the right of the xz-plane, and two units above the xy-plane.
2. The point is located on the y-axis, five units to the left of the xz-plane.
3. Find the distance between the points $(-2, 3, -6)$ and $(4, -5, 1)$.
4. Find the lengths of the sides of the right triangle at the right. Show that these lengths satisfy the Pythagorean Theorem.

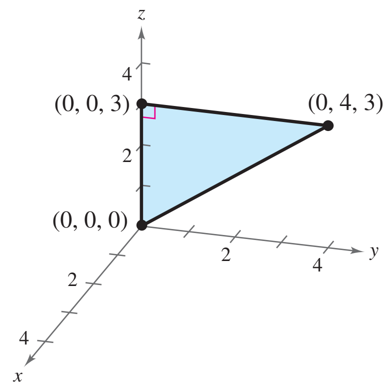


Figure for 4

5. Find the coordinates of the midpoint of the line segment joining $(3, 4, -1)$ and $(-5, 0, 2)$.
6. Find an equation of the sphere for which the endpoints of a diameter are $(0, 0, 0)$ and $(4, 4, 8)$.
7. Sketch the graph of the equation $(x - 2)^2 + (y + 1)^2 + z^2 = 4$, and then sketch the xy-trace and the yz-trace.
8. For the vectors $\mathbf{u} = \langle 2, -6, 0\rangle$ and $\mathbf{v} = \langle -4, 5, 3\rangle$, find $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{u} \times \mathbf{v}$.

In Exercises 9–11, determine whether u and v are orthogonal, parallel, or neither.

9. $\mathbf{u} = \langle 4, 4, 0\rangle$, $\mathbf{v} = \langle 0, -8, 6\rangle$
10. $\mathbf{u} = \langle 4, -2, 10\rangle$, $\mathbf{v} = \langle -2, 6, 2\rangle$
11. $\mathbf{u} = \langle -1, 6, -3\rangle$, $\mathbf{v} = \langle 3, -18, 9\rangle$

12. Find the volume of the parallelepiped with the vertices $A(1, 3, 2)$, $B(3, 4, 2)$, $C(3, 2, 2)$, $D(1, 1, 2)$, $E(1, 3, 5)$, $F(3, 4, 5)$, $G(3, 2, 5)$, and $H(1, 1, 5)$.
13. Find sets of (a) parametric equations and (b) symmetric equations for the line passing through the points $(-2, 3, 0)$ and $(5, 8, 25)$.
14. Find the parametric form of the equation of the line passing through the point $(-1, 2, 0)$ and perpendicular to $2x - 4y + z = 8$.
15. Find an equation of the plane passing through the points $(0, 0, 0)$, $(-2, 3, 0)$, and $(5, 8, 25)$.
16. Label the intercepts and sketch the graph of the plane given by $3x - 6y - 12z = 24$.
17. Find the distance between the point $(0, 0, 25)$ and the plane $2x - 5y + z = 10$.
18. A plastic wastebasket has the shape and dimensions shown in the figure. In fabricating a mold for making the wastebasket, it is necessary to know the angle between two adjacent sides. Find the angle.

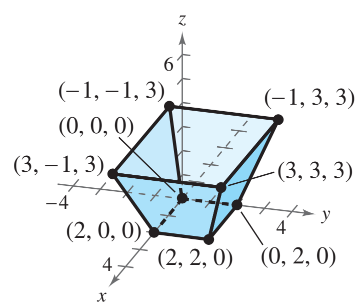


Figure for 18

In Exercises 19–27, find the limit (if it exists). If the limit does not exist, explain why. Use a graphing utility to verify your result graphically.

19. $\displaystyle\lim_{x\to 4}(5x - x^2)$
20. $\displaystyle\lim_{x\to -2^+}\frac{x + 2}{x^2 + x - 2}$
21. $\displaystyle\lim_{x\to 7}\frac{x - 7}{x^2 - 49}$
22. $\displaystyle\lim_{x\to 0}\frac{\sqrt{x + 4} - 2}{x}$
23. $\displaystyle\lim_{x\to 4^-}\frac{|x - 4|}{x - 4}$
24. $\displaystyle\lim_{x\to 0}\sin\left(\frac{\pi}{x}\right)$
25. $\displaystyle\lim_{x\to 0}\frac{\frac{1}{x + 3} - \frac{1}{3}}{x}$
26. $\displaystyle\lim_{x\to 0}\frac{\sqrt{x + 16} - 4}{x}$
27. $\displaystyle\lim_{x\to 2^-}\frac{x - 2}{x^2 - 4}$

In Exercises 28–31, find a formula for the slope of the graph of f at the point $(x, f(x))$. Then use it to find the slope at the specified point.

28. $f(x) = 4 - x^2, \quad (-2, 0)$

29. $f(x) = \sqrt{x + 3}, \quad (-2, 1)$

30. $f(x) = \dfrac{1}{x + 3}, \quad \left(1, \dfrac{1}{4}\right)$

31. $f(x) = x^2 - x, \quad (1, 0)$

In Exercises 32–37, find the limit (if it exists). If the limit does not exist, explain why. Use a graphing utility to verify your result graphically.

32. $\displaystyle\lim_{x\to\infty} \frac{x^3}{x^2 - 9}$

33. $\displaystyle\lim_{x\to\infty} \frac{3 - 7x}{x + 4}$

34. $\displaystyle\lim_{x\to\infty} \frac{3x^2 + 1}{x^2 + 4}$

35. $\displaystyle\lim_{x\to\infty} \frac{2x}{x^2 + 3x - 2}$

36. $\displaystyle\lim_{x\to\infty} \frac{3 - x}{x^2 + 1}$

37. $\displaystyle\lim_{x\to\infty} \frac{3 + 4x - x^3}{2x^2 + 3}$

In Exercises 38–40, evaluate the sum using the summation formulas and properties.

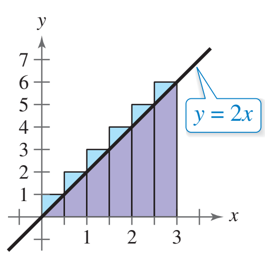
38. $\displaystyle\sum_{i=1}^{50} (1 - i^2)$

39. $\displaystyle\sum_{k=1}^{20} (3k^2 - 2k)$

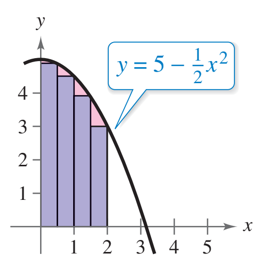
40. $\displaystyle\sum_{i=1}^{40} (12 + i^3)$

In Exercises 41–44, approximate the area of the region using the indicated number of rectangles of equal width.

41.



42.



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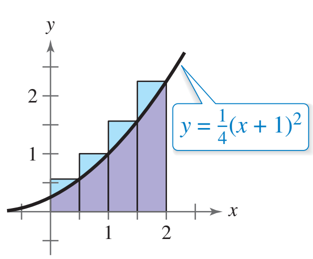

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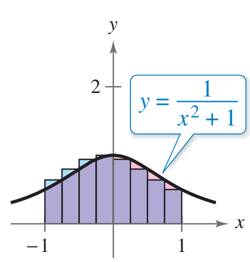

In Exercises 45–48, use the limit process to find the area of the region bounded by the graph of the function and the x-axis over the specified interval.

45. $f(x) = x + 2$
Interval: $[0, 1]$

46. $f(x) = x^2 + 1$
Interval: $[0, 4]$

47. $f(x) = 4 - x^2$
Interval: $[0, 2]$

48. $f(x) = 1 - x^3$
Interval: $[0, 1]$

Proofs in Mathematics

Many of the proofs of the definitions and properties presented in this chapter are beyond the scope of this text. Included below are simple proofs for the limit of a power function and the limit of a polynomial function.

Limit of a Power Function (p. 755)

$\lim_{x \to c} x^n = c^n$, c is a real number and n is a positive integer.

Proof

$$\lim_{x \to c} x^n = \lim_{x \to c} (\underbrace{x \cdot x \cdot x \cdot \cdot \cdot \cdot \cdot x}_{n \text{ factors}})$$

$$= \underbrace{\lim_{x \to c} x \cdot \lim_{x \to c} x \cdot \lim_{x \to c} x \cdot \cdot \cdot \cdot \cdot \lim_{x \to c} x}_{n \text{ factors}}$$ Product Property of Limits

$$= \underbrace{c \cdot c \cdot c \cdot \cdot \cdot \cdot \cdot c}_{n \text{ factors}}$$ Limit of the identity function

$$= c^n$$ Exponential form

Limit of a Polynomial Function (p. 756)

If p is a polynomial function and c is a real number, then

$$\lim_{x \to c} p(x) = p(c).$$

Proof

Let p be a polynomial function such that

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2 x^2 + a_1 x + a_0.$$

Because a polynomial function is the sum of monomial functions, you can write the following.

$$\lim_{x \to c} p(x) = \lim_{x \to c} (a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2 x^2 + a_1 x + a_0)$$

$$= \lim_{x \to c} a_n x^n + \lim_{x \to c} a_{n-1}x^{n-1} + \cdot \cdot \cdot + \lim_{x \to c} a_2 x^2 + \lim_{x \to c} a_1 x + \lim_{x \to c} a_0$$

$$= a_n c^n + a_{n-1}c^{n-1} + \cdot \cdot \cdot + a_2 c^2 + a_1 c + a_0$$ Scalar Multiple Property of Limits and limit of a power function

$$= p(c)$$ p evaluated at c

Proving Limits

To prove most of the definitions and properties from this chapter, you must use the *formal* definition of a limit. This definition is called the *epsilon-delta definition* and was first introduced by Karl Weierstrass (1815–1897). If you go on to take a course in calculus, you will use this definition of a limit extensively.