Section 3.2 Polynomial Functions and Their Graphs

Polynomial Functions

A **polynomial function of degree *n*** is a function of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where n is a nonnegative integer and $a_n \neq 0$.

The numbers $a_0, a_1, a_2, \ldots, a_n$ are called the **coefficients** of the polynomial.

The number a_0 is the **constant coefficient** or **constant term**.

The number a_n, the coefficient of the highest power, is the **leading coefficient**, and the term $a_n x^n$ is the **leading term**.

EXAMPLES:

$$P(x) = 3, \quad Q(x) = 4x - 7, \quad R(x) = x^2 + x, \quad S(x) = 2x^3 - 6x^2 - 10$$

QUESTION: Which of the following are polynomial functions?

(a) $f(x) = -x^3 + 2x + 4$

(b) $f(x) = (\sqrt{x})^3 - 2(\sqrt{x})^2 + 5(\sqrt{x}) - 1$

(c) $f(x) = (x-2)(x-1)(x+4)^2$

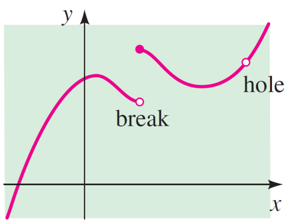
(d) $f(x) = \dfrac{x^2+2}{x^2-2}$

Answer: (a) and (c)

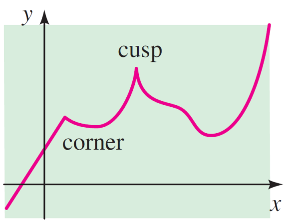
If a polynomial consists of just a single term, then it is called a **monomial**. For example, $P(x) = x^3$ and $Q(x) = -6x^5$ are monomials.

Graphs of Polynomials

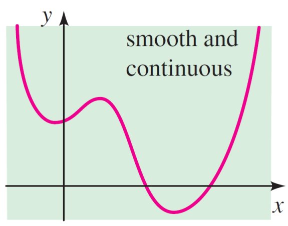
The graph of a polynomial function is always a smooth curve; that is, it has no breaks or corners.



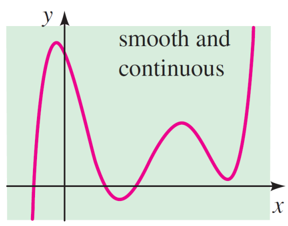
Not the graph of a polynomial function



Not the graph of a polynomial function

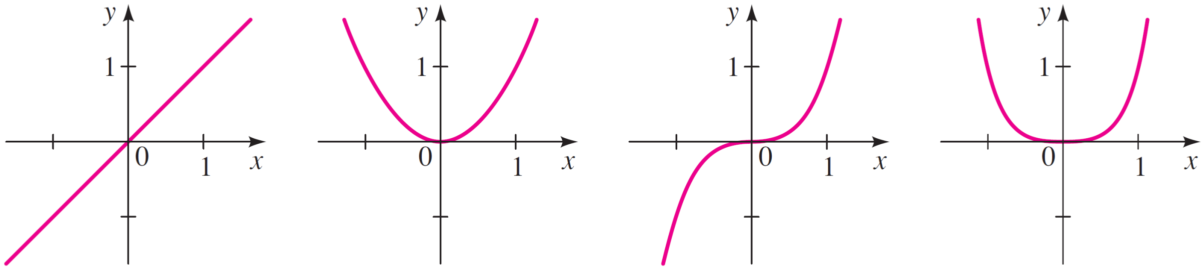


Graph of a polynomial function

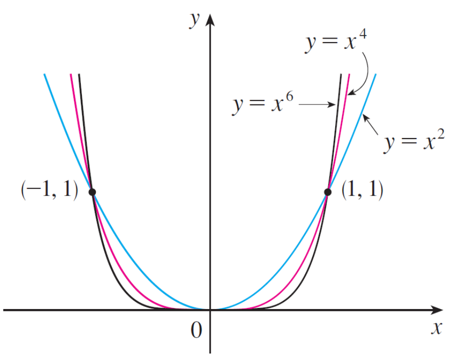


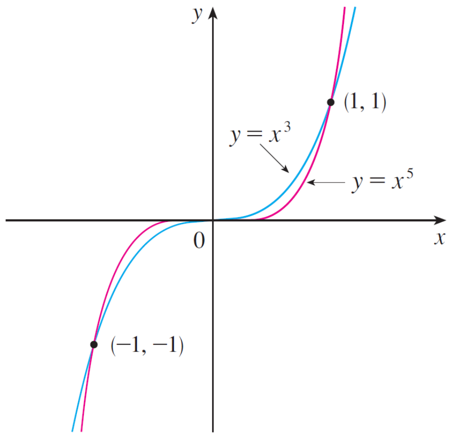
Graph of a polynomial function

The simplest polynomial functions are the monomials $P(x) = x^n$, whose graphs are shown in the Figure below.



(a) $y = x$ (b) $y = x^2$ (c) $y = x^3$ (d) $y = x^4$





EXAMPLE: Sketch the graphs of the following functions.

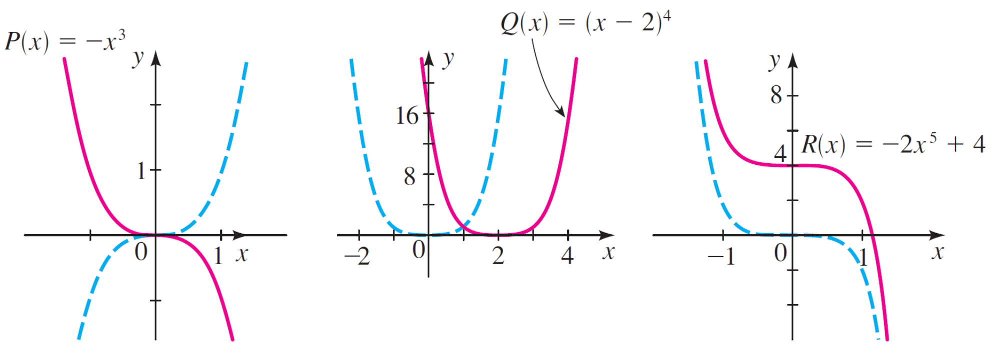
(a) $P(x) = -x^3$ (b) $Q(x) = (x-2)^4$ (c) $R(x) = -2x^5 + 4$

Solution:

(a) The graph of $P(x) = -x^3$ is the reflection of the graph of $y = x^3$ in the x-axis.

(b) The graph of $Q(x) = (x-2)^4$ is the graph of $y = x^4$ shifted to the right 2 units.

(c) We begin with the graph of $y = x^5$. The graph of $y = -2x^5$ is obtained by stretching the graph vertically and reflecting it in the x-axis. Finally, the graph of $R(x) = -2x^5 + 4$ is obtained by shifting upward 4 units.



(a) (b) (c)

EXAMPLE: Sketch the graphs of the following functions.

(a) $P(x) = -x^2$ (b) $Q(x) = (x+1)^5$ (c) $R(x) = -3x^2 + 3$

EXAMPLE: Sketch the graphs of the following functions.

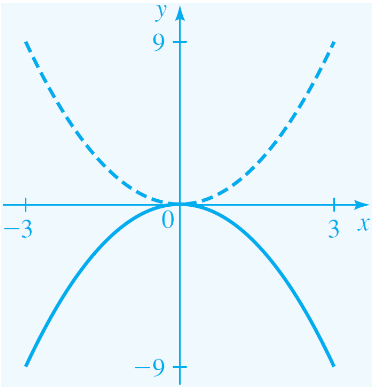
(a) $P(x)=-x^2$ (b) $Q(x)=(x+1)^5$ (c) $R(x)=-3x^2+3$

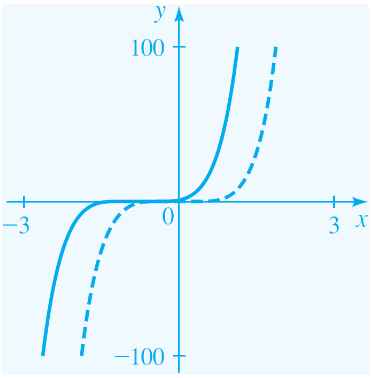
Solution:

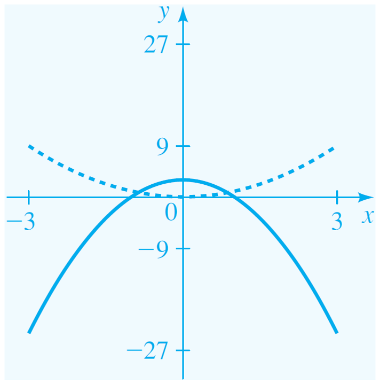
(a) The graph of $P(x)=-x^2$ is the reflection of the graph of $y=x^2$ in the x-axis.

(b) The graph of $Q(x)=(x+1)^5$ is the graph of $y=x^5$ shifted to the left 1 unit.

(c) We begin with the graph of $y=x^2$. The graph of $y=-3x^2$ is obtained by stretching the graph vertically and reflecting it in the x-axis. Finally, the graph of $R(x)=-3x^2+3$ is obtained by shifting upward 3 units.







End Behavior and the Leading Term

The **end behavior** of a polynomial is a description of what happens as x becomes large in the positive or negative direction. To describe end behavior, we use the following notation:

$x\to\infty$	means	"x becomes large in the positive direction"

$x\to-\infty$	means	"x becomes large in the negative direction"

For example, the monomial $y=x^2$ has the following end behavior:

$$y\to\infty \text{ as } x\to\infty \quad \text{and} \quad y\to\infty \text{ as } x\to-\infty$$

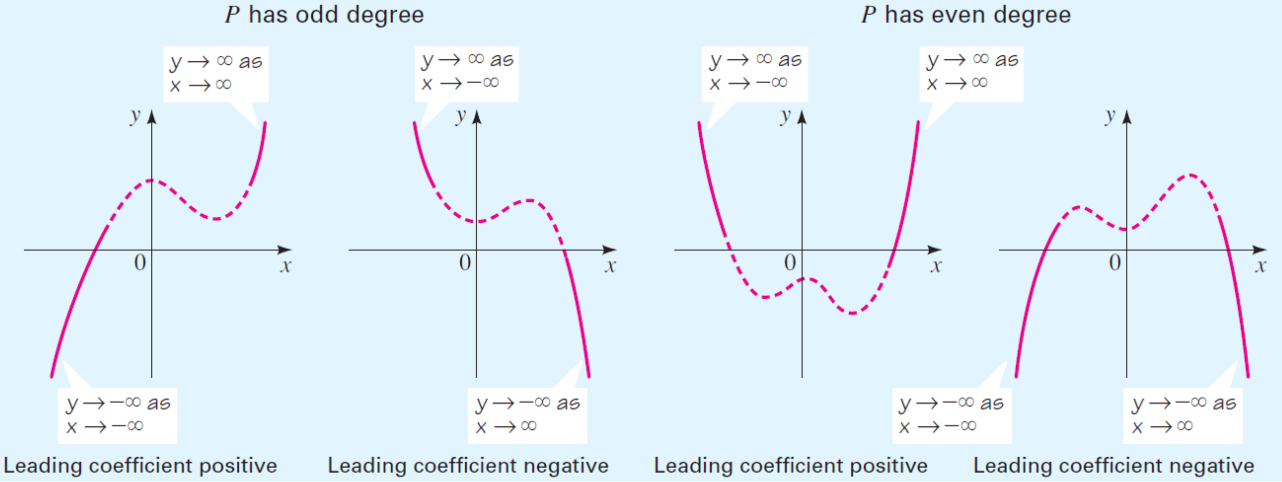
The monomial $y=x^3$ has the following end behavior:

$$y\to\infty \text{ as } x\to\infty \quad \text{and} \quad y\to-\infty \text{ as } x\to-\infty$$

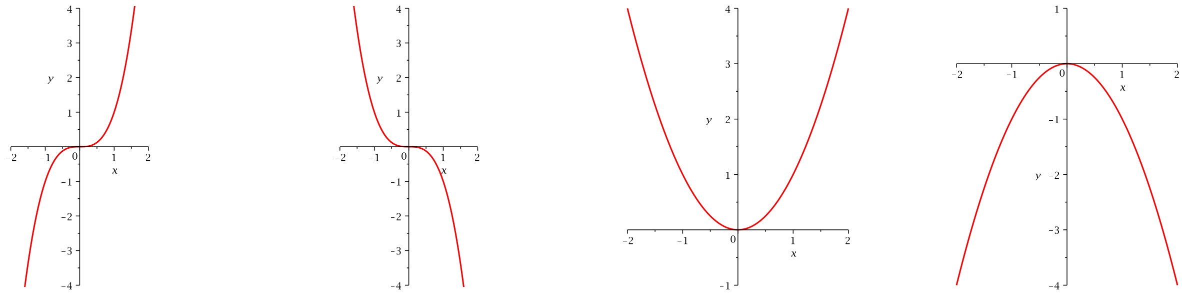
For any polynomial, *the end behavior is determined by the term that contains the highest power of x,* because when x is large, the other terms are relatively insignificant in size.

End Behavior of Polynomials

The end behavior of the polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is determined by the degree n and the sign of the leading coefficient a_n, as indicated in the following graphs.



COMPARE: Here are the graphs of the monomials x^3, $-x^3$, x^2, and $-x^2$.



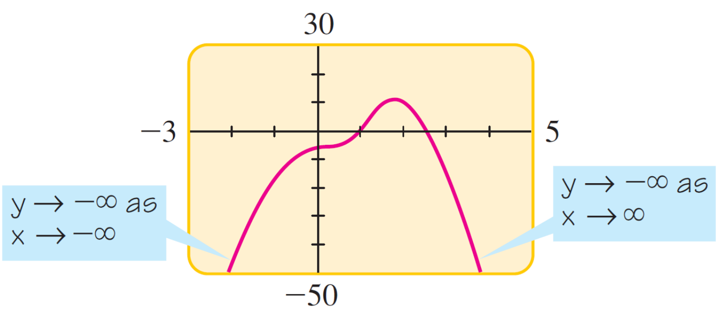
EXAMPLE: Determine the end behavior of the polynomial

$$P(x) = -2x^4 + 5x^3 + 4x - 7$$

Solution: The polynomial P has degree 4 and leading coefficient -2. Thus, P has *even* degree and *negative* leading coefficient, so it has the following end behavior:

$$y \to -\infty \text{ as } x \to \infty \quad \text{and} \quad y \to -\infty \text{ as } x \to -\infty$$

The graph in the Figure below illustrates the end behavior of P.



EXAMPLE: Determine the end behavior of the polynomial

$$P(x) = -3x^3 + 20x^2 + 60x + 2$$

EXAMPLE: Determine the end behavior of the polynomial

$$P(x) = -3x^3 + 20x^2 + 60x + 2$$

Answer:

$$y \to -\infty \text{ as } x \to \infty \quad \text{and} \quad y \to \infty \text{ as } x \to -\infty$$

EXAMPLE: Determine the end behavior of the polynomial

$$P(x) = 8x^3 - 7x^2 + 3x + 7$$

Answer:

$$y \to -\infty \text{ as } x \to -\infty \quad \text{and} \quad y \to \infty \text{ as } x \to \infty$$

EXAMPLE:

(a) Determine the end behavior of the polynomial $P(x) = 3x^5 - 5x^3 + 2x$.

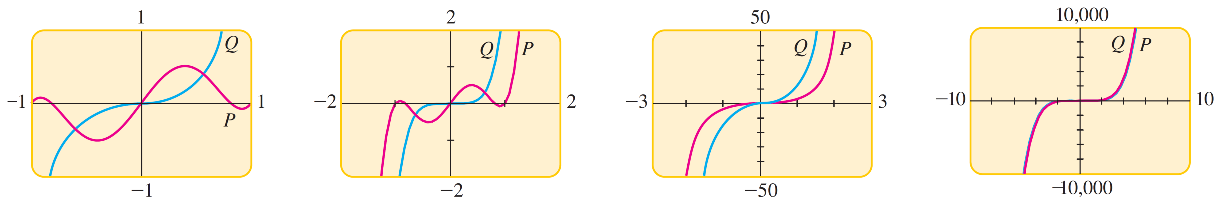
(b) Confirm that P and its leading term $Q(x) = 3x^5$ have the same end behavior by graphing them together.

Solution:

(a) Since P has odd degree and positive leading coefficient, it has the following end behavior:

$$y \to \infty \text{ as } x \to \infty \quad \text{and} \quad y \to -\infty \text{ as } x \to -\infty$$

(b) The Figure below shows the graphs of P and Q in progressively larger viewing rectangles. The larger the viewing rectangle, the more the graphs look alike. This confirms that they have the same end behavior.



To see algebraically why P and Q have the same end behavior, factor P as follows and compare with Q.

$$P(x) = 3x^5\left(1 - \frac{5}{3x^2} + \frac{2}{3x^4}\right) \qquad\qquad Q(x) = 3x^5$$

When x is large, the terms $5/3x^2$ and $2/3x^4$ are close to 0. So for large x, we have

$$P(x) \approx 3x^5(1 - 0 + 0) = 3x^5 = Q(x)$$

So when x is large, P and Q have approximately the same values.

By the same reasoning we can show that the end behavior of *any* polynomial is determined by its leading term.

Using Zeros to Graph Polynomials

If P is a polynomial function, then c is called a **zero** of P if $P(c) = 0$. In other words, the zeros of P are the solutions of the polynomial equation $P(x) = 0$. Note that if $P(c) = 0$, then the graph of P has an x-intercept at $x = c$, so the x-intercepts of the graph are the zeros of the function.

Real Zeros of Polynomials

If P is a polynomial and c is a real number, then the following are equivalent.

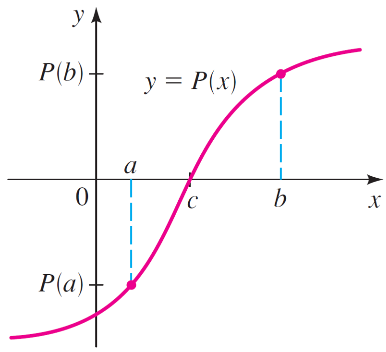
1. c is a zero of P.
2. $x = c$ is a solution of the equation $P(x) = 0$.
3. $x - c$ is a factor of $P(x)$.
4. $x = c$ is an x-intercept of the graph of P.

The following theorem has many important consequences.

Intermediate Value Theorem for Polynomials

If P is a polynomial function and $P(a)$ and $P(b)$ have opposite signs, then there exists at least one value c between a and b for which $P(c) = 0$.

One important consequence of this theorem is that between any two successive zeros, the values of a polynomial are either all positive or all negative. This observation allows us to use the following guidelines to graph polynomial functions.

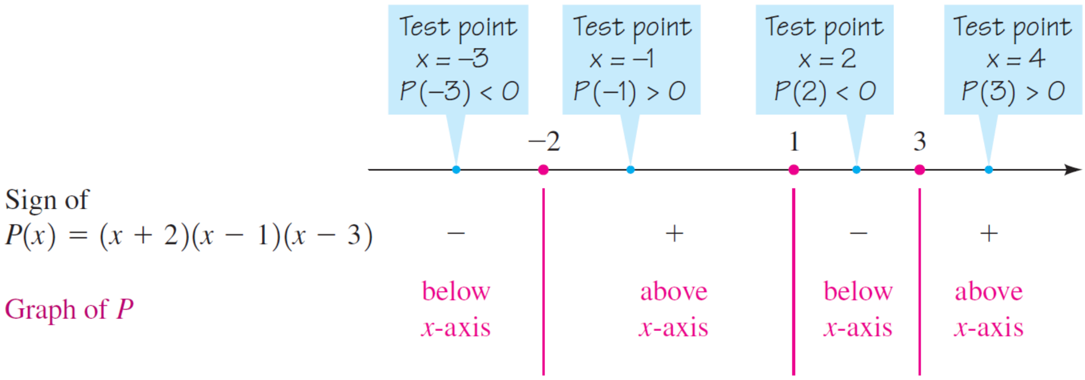


Guidelines for Graphing Polynomial Functions

1. **Zeros.** Factor the polynomial to find all its real zeros; these are the x-intercepts of the graph.
2. **Test Points.** Make a table of values for the polynomial. Include test points to determine whether the graph of the polynomial lies above or below the x-axis on the intervals determined by the zeros. Include the y-intercept in the table.
3. **End Behavior.** Determine the end behavior of the polynomial.
4. **Graph.** Plot the intercepts and other points you found in the table. Sketch a smooth curve that passes through these points and exhibits the required end behavior.

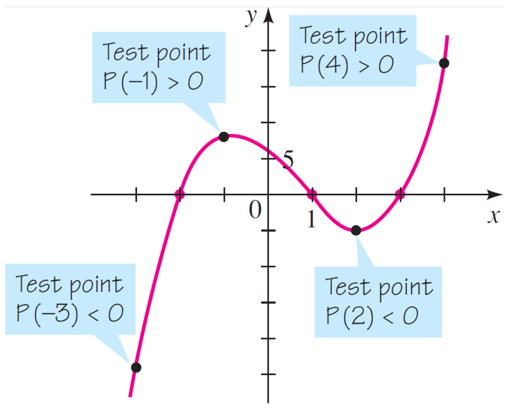
EXAMPLE: Sketch the graph of the polynomial function $P(x) = (x+2)(x-1)(x-3)$.

Solution: The zeros are $x = -2, 1$, and 3. These determine the intervals $(-\infty, -2), (-2, 1), (1, 3)$, and $(3, \infty)$. Using test points in these intervals, we get the information in the following sign diagram.



Plotting a few additional points and connecting them with a smooth curve helps us complete the graph.

	x	$P(x)$
Test point →	-3	-24
	-2	0
Test point →	-1	8
	0	6
	1	0
Test point →	2	-4
	3	0
Test point →	4	18

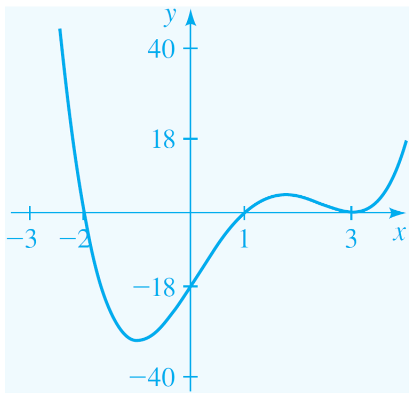


EXAMPLE: Sketch the graph of the polynomial function $P(x) = (x+2)(x-1)(x-3)^2$.

Solution: The zeros are $-2, 1$, and 3. End term behavior:

$$y \to \infty \text{ as } x \to \infty \quad \text{and} \quad y \to \infty \text{ as } x \to -\infty$$

We use test points 0 and 2 to obtain the graph:



EXAMPLE: Let $P(x) = x^3 - 2x^2 - 3x$.

(a) Find the zeros of P.

(b) Sketch the graph of P.

EXAMPLE: Let $P(x) = x^3 - 2x^2 - 3x$.

(a) Find the zeros of P. (b) Sketch the graph of P.

Solution:

(a) To find the zeros, we factor completely:

$$\begin{aligned} P(x) &= x^3 - 2x^2 - 3x \\ &= x(x^2 - 2x - 3) \\ &= x(x-3)(x+1) \end{aligned}$$

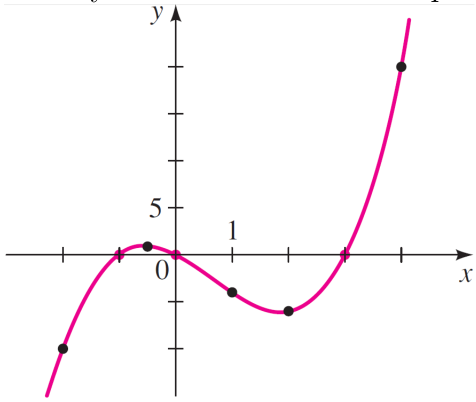
Thus, the zeros are $x = 0,\ x = 3$, and $x = -1$.

(b) The x-intercepts are $x = 0,\ x = 3$, and $x = -1$. The y-intercept is $P(0) = 0$. We make a table of values of $P(x)$, making sure we choose test points between (and to the right and left of) successive zeros. Since P is of odd degree and its leading coefficient is positive, it has the following end behavior:

$$y \to \infty \text{ as } x \to \infty \quad \text{and} \quad y \to -\infty \text{ as } x \to -\infty$$

We plot the points in the table and connect them by a smooth curve to complete the graph.

	x	$P(x)$
	-2	-10
	-1	0
Test point →	$-\frac{1}{2}$	$\frac{7}{8}$
	0	0
Test point →	1	-4
	2	-6
	3	0
	4	20



EXAMPLE: Let $P(x) = x^3 - 9x^2 + 20x$.

(a) Find the zeros of P. (b) Sketch the graph of P.

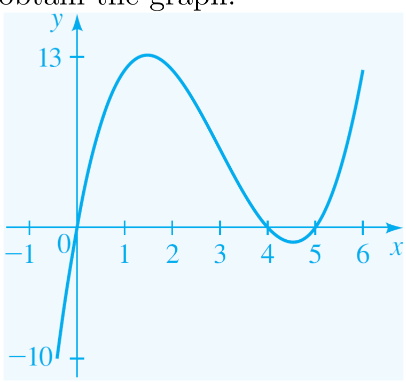
Solution:

(a) $P(x) = x(x-4)(x-5)$, so the zeros are $x = 0,\ x = 4,\ x = 5$.

(b) End term behavior:

$$y \to \infty \text{ as } x \to \infty \quad \text{and} \quad y \to -\infty \text{ as } x \to -\infty$$

We use test points 3 and 4.5 to obtain the graph:



EXAMPLE: Let $P(x) = -2x^4 - x^3 + 3x^2$.

(a) Find the zeros of P. (b) Sketch the graph of P.

Solution:

(a) To find the zeros, we factor completely:

$$P(x) = -2x^4 - x^3 + 3x^2 = -x^2(2x^2 + x - 3) = -x^2(2x+3)(x-1)$$

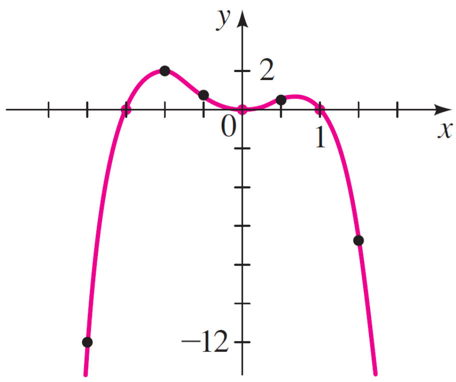
Thus, the zeros are $x = 0,\ x = -\frac{3}{2}$, and $x = 1$.

(b) The x-intercepts are $x = 0,\ x = -\frac{3}{2}$, and $x = 1$. The y-intercept is $P(0) = 0$. We make a table of values of $P(x)$, making sure we choose test points between (and to the right and left of) successive zeros. Since P is of even degree and its leading coefficient is negative, it has the following end behavior:

$$y \to -\infty \text{ as } x \to \infty \quad \text{and} \quad y \to -\infty \text{ as } x \to -\infty$$

We plot the points in the table and connect them by a smooth curve to complete the graph.

x	$P(x)$
-2	-12
-1.5	0
-1	2
-0.5	0.75
0	0
0.5	0.5
1	0
1.5	-6.75



EXAMPLE: Let $P(x) = 3x^4 - 5x^3 - 12x^2$.

(a) Find the zeros of P. (b) Sketch the graph of P.

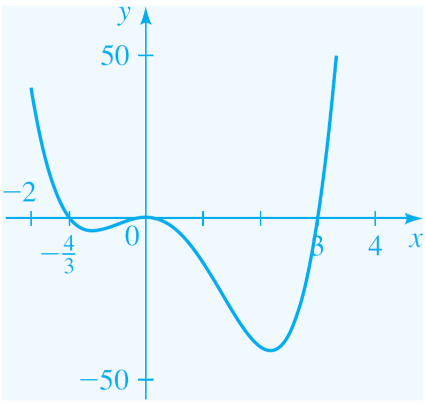
Solution:

(a) $P(x) = x^2(x-3)(3x+4)$, so the zeros are $x = 0,\ x = 3,\ x = -4/3$.

(b) End term behavior:

$$y \to \infty \text{ as } x \to \infty \quad \text{and} \quad y \to -\infty \text{ as } x \to -\infty$$

We use test points -1 and 1 to obtain the graph:



EXAMPLE: Let $P(x) = x^3 - 2x^2 - 4x + 8$.

(a) Find the zeros of P. (b) Sketch the graph of P.

Solution:

(a) To find the zeros, we factor completely:

$$\begin{aligned} P(x) = x^3 - 2x^2 - 4x + 8 = x^2(x-2) - 4(x-2) &= (x^2-4)(x-2) \\ &= (x+2)(x-2)(x-2) \\ &= (x+2)(x-2)^2 \end{aligned}$$

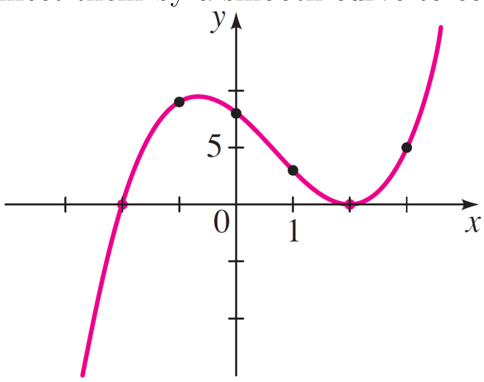
Thus the zeros are $x = -2$ and $x = 2$.

(b) The x-intercepts are $x = -2$ and $x = 2$. The y-intercept is $P(0) = 8$. The table gives additional values of $P(x)$. Since P is of odd degree and its leading coefficient is positive, it has the following end behavior:

$$y \to \infty \text{ as } x \to \infty \quad \text{and} \quad y \to -\infty \text{ as } x \to -\infty$$

We plot the points in the table and connect them by a smooth curve to complete the graph.

x	$P(x)$
−3	−25
−2	0
−1	9
0	8
1	3
2	0
3	5



EXAMPLE: Let $P(x) = x^3 + 3x^2 - 9x - 27$.

(a) Find the zeros of P. (b) Sketch the graph of P.

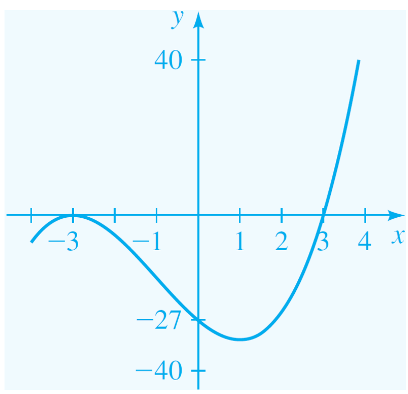
Answer:

(a) $P(x) = (x+3)^2(x-3)$, so the zeros are $x = -3,\ x = 3$.

(b) End term behavior:

$$y \to \infty \text{ as } x \to \infty \quad \text{and} \quad y \to -\infty \text{ as } x \to -\infty$$

We use test point 0 to obtain the graph:

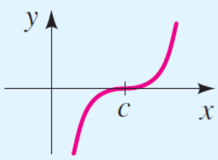
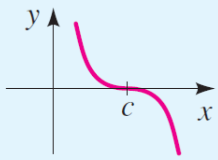
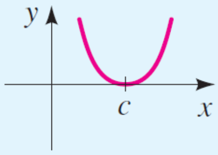
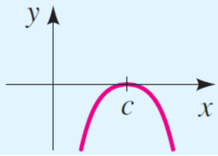


Shape of the Graph Near a Zero

If c is a zero of P and the corresponding factor $x-c$ occurs exactly m times in the factorization of P then we say that c is a **zero of multiplicity m.** One can show that the graph of P crosses the x-axis at c if the multiplicity m is odd and does not cross the x-axis if m is even. Moreover, it can be shown that near $x=c$ the graph has the same general shape as $y=A(x-c)^m$.

Shape of the Graph Near a Zero of Multiplicity m

Suppose that c is a zero of P of multiplicity m. Then the shape of the graph of P near c is as follows.

Multiplicity of c	Shape of the graph of P near the x-intercept c
m odd, $m>1$	 OR 
m even, $m>1$	 OR 

EXAMPLE: Graph the polynomial $P(x)=x^4(x-2)^3(x+1)^2$.

Solution: The zeros of P are $-1, 0$, and 2, with multiplicities 2, 4, and 3, respectively.

0 is a zero of multiplicity 4.

2 is a zero of multiplicity 3.

-1 is a zero of multiplicity 2.

$$P(x) = x^4(x-2)^3(x+1)^2$$

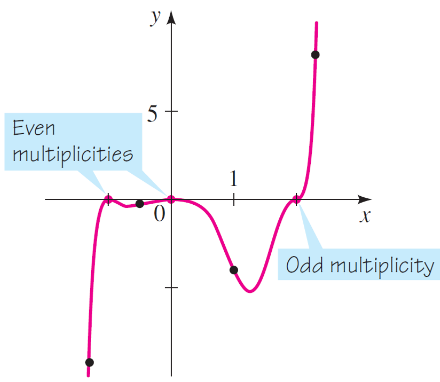
The zero 2 has *odd* multiplicity, so the graph crosses the x-axis at the x-intercept 2. But the zeros 0 and -1 have *even* multiplicity, so the graph does not cross the x-axis at the x-intercepts 0 and -1.

Since P is a polynomial of degree 9 and has positive leading coefficient, it has the following end behavior:

$$y \to \infty \text{ as } x \to \infty \quad \text{and} \quad y \to -\infty \text{ as } x \to -\infty$$

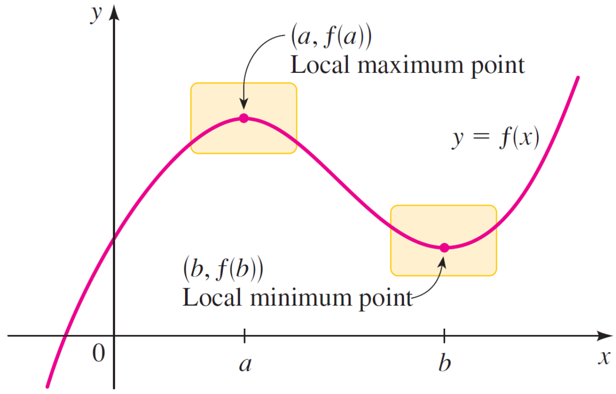
With this information and a table of values, we sketch the graph.

x	$P(x)$
-1.3	-9.2
-1	0
-0.5	-3.9
0	0
1	-4
2	0
2.3	8.2



Local Maxima and Minima of Polynomials

If the point $(a, f(a))$ is the highest point on the graph of f within some viewing rectangle, then $(a, f(a))$ is a **local maximum point** on the graph and if $(b, f(b))$ is the lowest point on the graph of f within some viewing rectangle, then $(b, f(b))$ is a **local minimum point**. The set of all local maximum and minimum points on the graph of a function is called its **local extrema**.



For a polynomial function the number of local extrema must be less than the degree, as the following principle indicates.

Local Extrema of Polynomials

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial of degree n, then the graph of P has at most $n - 1$ local extrema.

A polynomial of degree n may in fact have less than $n-1$ local extrema. For example, $P(x) = x^3$ has no local extrema, even though it is of degree 3.

EXAMPLE: Determine how many local extrema each polynomial has.

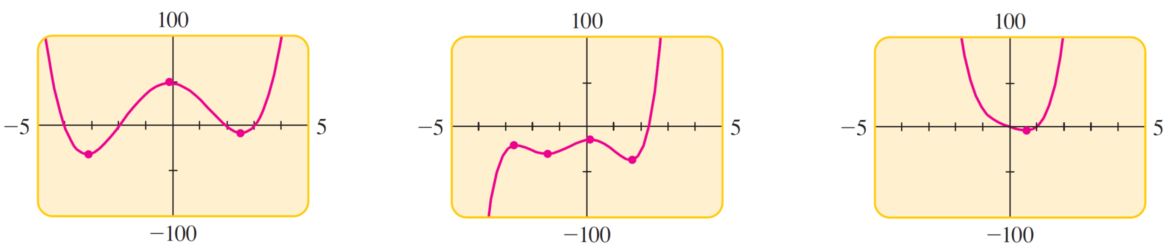
(a) $P_1(x) = x^4 + x^3 - 16x^2 - 4x + 48$ (b) $P_2(x) = x^5 + 3x^4 - 5x^3 - 15x^2 + 4x - 15$
(c) $P_3(x) = 7x^4 + 3x^2 - 10x$

Solution:

(a) P_1 has two local minimum points and one local maximum point, for a total of three local extrema.

(b) P_2 has two local minimum points and two local maximum points, for a total of four local extrema.

(c) P_3 has just one local extremum, a local minimum.



(a) $P_1(x) = x^4 + x^3 - 16x^2 - 4x + 48$

(b) $P_2(x) = x^5 + 3x^4 - 5x^3 - 15x^2 + 4x - 15$

(c) $P_3(x) = 7x^4 + 3x^2 - 10x$

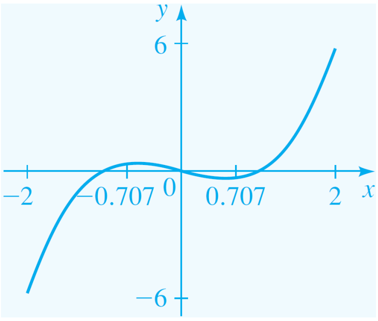
EXAMPLE: Determine how many local extrema each polynomial has.

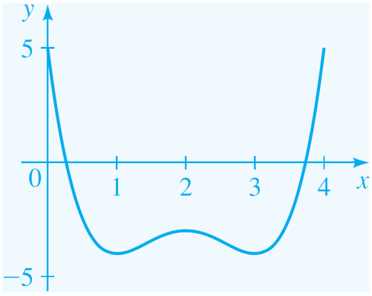
(a) $P_1(x) = x^3 - x$ (b) $P_2(x) = x^4 - 8x^3 + 22x^2 - 24x + 5$

Solution:

(a) P_1 has one local minimum point and one local maximum point for a total of two local extrema.

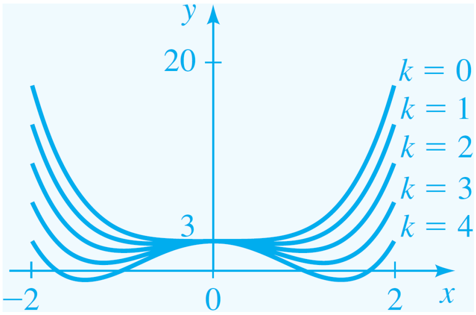
(b) P_2 has two local minimum points and one local maximum point for a total of three local extrema.





EXAMPLE: Sketch the family of polynomials $P(x) = x^4 - kx^2 + 3$ for $k = 0, 1, 2, 3$, and 4. How does changing the value of k affect the graph?

Solution: The polynomials are graphed below. We see that increasing the value of k causes the two local minima to dip lower and lower.



EXAMPLE: Sketch the family of polynomials $P(x) = x^3 - cx^2$ for $c = 0, 1, 2$, and 3. How does changing the value of c affect the graph?

EXAMPLE: Sketch the family of polynomials $P(x) = x^3 - cx^2$ for $c = 0, 1, 2$, and 3. How does changing the value of c affect the graph?

Solution: The polynomials

$$P_0(x) = x^3, \quad P_1(x) = x^3 - x^2, \quad P_2(x) = x^3 - 2x^2, \quad P_3(x) = x^3 - 3x^2$$

are graphed in the Figure below. We see that increasing the value of c causes the graph to develop an increasingly deep "valley" to the right of the y-axis, creating a local maximum at the origin and a local minimum at a point in quadrant IV. This local minimum moves lower and farther to the right as c increases. To see why this happens, factor $P(x) = x^2(x - c)$. The polynomial P has zeros at 0 and c, and the larger c gets, the farther to the right the minimum between 0 and c will be.