EXAMPLES:

\[ P(x) = 3; \quad Q(x) = 4x^7; \quad R(x) = x^2 + x; \quad S(x) = 2x^3 - 6x^2 - 10 \]

QUESTION: Which of the following are polynomial functions?

(a) \( f(x) = -x^3 + 2x + 4 \)

(b) \( f(x) = (\sqrt{x})^3 - 2(\sqrt{x})^2 + 5(\sqrt{x}) - 1 \)

(c) \( f(x) = (x - 2)(x - 1)(x + 4)^2 \)

(d) \( f(x) = \frac{x^2 + 2}{x^2 - 2} \)

Answer: (a) and (c)

If a polynomial consists of just a single term, then it is called a **monomial**. For example, \( P(x) = x^3 \) and \( Q(x) = -6x^5 \) are monomials.

**Graphs of Polynomials**

The graph of a polynomial function is always a smooth curve; that is, it has no breaks or corners.
The simplest polynomial functions are the monomials \( P(x) = x^n \), whose graphs are shown in the Figure below.

**EXAMPLE:** Sketch the graphs of the following functions.

(a) \( P(x) = -x^3 \)  
(b) \( Q(x) = (x - 2)^4 \)  
(c) \( R(x) = -2x^5 + 4 \)

**Solution:**

(a) The graph of \( P(x) = -x^3 \) is the reflection of the graph of \( y = x^3 \) in the \( x \)-axis.

(b) The graph of \( Q(x) = (x - 2)^4 \) is the graph of \( y = x^4 \) shifted to the right 2 units.

(c) We begin with the graph of \( y = x^5 \). The graph of \( y = -2x^5 \) is obtained by stretching the graph vertically and reflecting it in the \( x \)-axis. Finally, the graph of \( R(x) = -2x^5 + 4 \) is obtained by shifting upward 4 units.

**EXAMPLE:** Sketch the graphs of the following functions.

(a) \( P(x) = -x^2 \)  
(b) \( Q(x) = (x + 1)^5 \)  
(c) \( R(x) = -3x^2 + 3 \)

2
EXAMPLE: Sketch the graphs of the following functions.

(a) \( P(x) = -x^2 \)

(b) \( Q(x) = (x + 1)^5 \)

(c) \( R(x) = -3x^2 + 3 \)

Solution:

(a) The graph of \( P(x) = -x^2 \) is the reflection of the graph of \( y = x^2 \) in the \( x \)-axis.

(b) The graph of \( Q(x) = (x + 1)^5 \) is the graph of \( y = x^5 \) shifted to the left 1 unit.

(c) We begin with the graph of \( y = x^2 \). The graph of \( y = -3x^2 \) is obtained by stretching the graph vertically and reflecting it in the \( x \)-axis. Finally, the graph of \( R(x) = -3x^2 + 3 \) is obtained by shifting upward 3 units.

End Behavior and the Leading Term

The end behavior of a polynomial is a description of what happens as \( x \) becomes large in the positive or negative direction. To describe end behavior, we use the following notation:

\[
\begin{align*}
  x & \to \infty \quad \text{means} \quad \text{“}x \text{ becomes large in the positive direction”} \\
  x & \to -\infty \quad \text{means} \quad \text{“}x \text{ becomes large in the negative direction”}
\end{align*}
\]

For example, the monomial \( y = x^2 \) has the following end behavior:

\[
y \to \infty \text{ as } x \to \infty \quad \text{and} \quad y \to \infty \text{ as } x \to -\infty
\]

The monomial \( y = x^3 \) has the following end behavior:

\[
y \to \infty \text{ as } x \to \infty \quad \text{and} \quad y \to -\infty \text{ as } x \to -\infty
\]

For any polynomial, the end behavior is determined by the term that contains the highest power of \( x \), because when \( x \) is large, the other terms are relatively insignificant in size.
COMPARE: Here are the graphs of the monomials $x^3$, $-x^3$, $x^2$, and $-x^2$.

EXAMPLE: Determine the end behavior of the polynomial 

$$P(x) = -2x^4 + 5x^3 + 4x - 7$$

Solution: The polynomial $P$ has degree 4 and leading coefficient $-2$. Thus, $P$ has even degree and negative leading coefficient, so it has the following end behavior:

$$y \rightarrow -\infty \text{ as } x \rightarrow \infty \; \text{ and } \; y \rightarrow -\infty \text{ as } x \rightarrow -\infty$$

The graph in the Figure below illustrates the end behavior of $P$.

EXAMPLE: Determine the end behavior of the polynomial

$$P(x) = -3x^3 + 20x^2 + 60x + 2$$
EXAMPLE: Determine the end behavior of the polynomial

\[ P(x) = -3x^3 + 20x^2 + 60x + 2 \]

Answer:

\[ y \to -\infty \text{ as } x \to \infty \quad \text{and} \quad y \to \infty \text{ as } x \to -\infty \]

EXAMPLE: Determine the end behavior of the polynomial

\[ P(x) = 8x^3 - 7x^2 + 3x + 7 \]

Answer:

\[ y \to -\infty \text{ as } x \to -\infty \quad \text{and} \quad y \to \infty \text{ as } x \to \infty \]

EXAMPLE:

(a) Determine the end behavior of the polynomial \( P(x) = 3x^5 - 5x^3 + 2x \).

(b) Confirm that \( P \) and its leading term \( Q(x) = 3x^5 \) have the same end behavior by graphing them together.

Solution:

(a) Since \( P \) has odd degree and positive leading coefficient, it has the following end behavior:

\[ y \to \infty \text{ as } x \to \infty \quad \text{and} \quad y \to -\infty \text{ as } x \to -\infty \]

(b) The Figure below shows the graphs of \( P \) and \( Q \) in progressively larger viewing rectangles. The larger the viewing rectangle, the more the graphs look alike. This confirms that they have the same end behavior.

To see algebraically why \( P \) and \( Q \) have the same end behavior, factor \( P \) as follows and compare with \( Q \).

\[ P(x) = 3x^5 \left(1 - \frac{5}{3x^2} + \frac{2}{3x^4}\right) \quad Q(x) = 3x^5 \]

When \( x \) is large, the terms \( 5/3x^2 \) and \( 2/3x^4 \) are close to 0. So for large \( x \), we have

\[ P(x) \approx 3x^5(1 - 0 + 0) = 3x^5 = Q(x) \]

So when \( x \) is large, \( P \) and \( Q \) have approximately the same values.

By the same reasoning we can show that the end behavior of any polynomial is determined by its leading term.
If \( P \) is a polynomial function, then \( c \) is called a **zero** of \( P \) if \( P(c) = 0 \). In other words, the zeros of \( P \) are the solutions of the polynomial equation \( P(x) = 0 \). Note that if \( P(c) = 0 \), then the graph of \( P \) has an \( x \)-intercept at \( x = c \), so the \( x \)-intercepts of the graph are the zeros of the function.

### Real Zeros of Polynomials

If \( P \) is a polynomial and \( c \) is a real number, then the following are equivalent.

1. \( c \) is a zero of \( P \).
2. \( x = c \) is a solution of the equation \( P(x) = 0 \).
3. \( x - c \) is a factor of \( P(x) \).
4. \( x = c \) is an \( x \)-intercept of the graph of \( P \).

The following theorem has many important consequences.

### Intermediate Value Theorem for Polynomials

If \( P \) is a polynomial function and \( P(a) \) and \( P(b) \) have opposite signs, then there exists at least one value \( c \) between \( a \) and \( b \) for which \( P(c) = 0 \).

One important consequence of this theorem is that between any two successive zeros, the values of a polynomial are either all positive or all negative. This observation allows us to use the following guidelines to graph polynomial functions.

### Guidelines for Graphing Polynomial Functions

1. **Zeros.** Factor the polynomial to find all its real zeros; these are the \( x \)-intercepts of the graph.

2. **Test Points.** Make a table of values for the polynomial. Include test points to determine whether the graph of the polynomial lies above or below the \( x \)-axis on the intervals determined by the zeros. Include the \( y \)-intercept in the table.

3. **End Behavior.** Determine the end behavior of the polynomial.

4. **Graph.** Plot the intercepts and other points you found in the table. Sketch a smooth curve that passes through these points and exhibits the required end behavior.
EXAMPLE: Sketch the graph of the polynomial function \( P(x) = (x + 2)(x - 1)(x - 3) \).

Solution: The zeros are \( x = -2, 1, \) and 3. These determine the intervals \((-\infty, -2), (-2, 1), (1, 3), \) and \((3, \infty)\). Using test points in these intervals, we get the information in the following sign diagram.

Plotting a few additional points and connecting them with a smooth curve helps us complete the graph.

EXAMPLE: Sketch the graph of the polynomial function \( P(x) = (x + 2)(x - 1)(x - 3)^2 \).

Solution: The zeros are \(-2, 1, \) and 3. End term behavior:

\[ y \to \infty \text{ as } x \to -\infty \quad \text{and} \quad y \to \infty \text{ as } x \to \infty \]

We use test points 0 and 2 to obtain the graph:

EXAMPLE: Let \( P(x) = x^3 - 2x^2 - 3x \).

(a) Find the zeros of \( P \). 

(b) Sketch the graph of \( P \).
EXAMPLE: Let \( P(x) = x^3 - 2x^2 - 3x \).

(a) Find the zeros of \( P \). 
(b) Sketch the graph of \( P \).

Solution:

(a) To find the zeros, we factor completely:

\[
P(x) = x^3 - 2x^2 - 3x \\
= x(x^2 - 2x - 3) \\
= x(x - 3)(x + 1)
\]

Thus, the zeros are \( x = 0 \), \( x = 3 \), and \( x = -1 \).

(b) The \( x \)-intercepts are \( x = 0 \), \( x = 3 \), and \( x = -1 \). The \( y \)-intercept is \( P(0) = 0 \). We make a table of values of \( P(x) \), making sure we choose test points between (and to the right and left of) successive zeros. Since \( P \) is of odd degree and its leading coefficient is positive, it has the following end behavior:

\[
y \to \infty \text{ as } x \to \infty \quad \text{and} \quad y \to -\infty \text{ as } x \to -\infty
\]

We plot the points in the table and connect them by a smooth curve to complete the graph.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( P(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-10</td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>-\frac{1}{2}</td>
<td>\frac{7}{8}</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>-4</td>
</tr>
<tr>
<td>2</td>
<td>-6</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>20</td>
</tr>
</tbody>
</table>

TEST point \( \to \)

EXAMPLE: Let \( P(x) = x^3 - 9x^2 + 20x \).

(a) Find the zeros of \( P \). 
(b) Sketch the graph of \( P \).

Solution:

(a) \( P(x) = x(x - 4)(x - 5) \), so the zeros are \( x = 0 \), \( x = 4 \), \( x = 5 \).

(b) End term behavior:

\[
y \to \infty \text{ as } x \to \infty \quad \text{and} \quad y \to -\infty \text{ as } x \to -\infty
\]

We use test points 3 and 4.5 to obtain the graph:
EXAMPLE: Let $P(x) = -2x^4 - x^3 + 3x^2$.

(a) Find the zeros of $P$. 
(b) Sketch the graph of $P$.

Solution:

(a) To find the zeros, we factor completely:

$$P(x) = -2x^4 - x^3 + 3x^2 = -x^2(2x^2 + x - 3) = -x^2(2x + 3)(x - 1)$$

Thus, the zeros are $x = 0$, $x = -\frac{3}{2}$, and $x = 1$.

(b) The $x$-intercepts are $x = 0$, $x = -\frac{3}{2}$, and $x = 1$. The $y$-intercept is $P(0) = 0$. We make a table of values of $P(x)$, making sure we choose test points between (and to the right and left of) successive zeros. Since $P$ is of even degree and its leading coefficient is negative, it has the following end behavior:

$$y \to -\infty \text{ as } x \to \infty \quad \text{and} \quad y \to -\infty \text{ as } x \to -\infty$$

We plot the points in the table and connect them by a smooth curve to complete the graph.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$P(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-12</td>
</tr>
<tr>
<td>-1.5</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>-0.5</td>
<td>0.75</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1.5</td>
<td>-6.75</td>
</tr>
</tbody>
</table>

EXAMPLE: Let $P(x) = 3x^4 - 5x^3 - 12x^2$.

(a) Find the zeros of $P$. 
(b) Sketch the graph of $P$.

Solution:

(a) $P(x) = x^2(x - 3)(3x + 4)$, so the zeros are $x = 0$, $x = 3$, $x = -4/3$.

(b) End term behavior:

$$y \to \infty \text{ as } x \to \infty \quad \text{and} \quad y \to -\infty \text{ as } x \to -\infty$$

We use test points $-1$ and $1$ to obtain the graph:
EXAMPLE: Let \( P(x) = x^3 - 2x^2 - 4x + 8 \).
(a) Find the zeros of \( P \).
(b) Sketch the graph of \( P \).

Solution:
(a) To find the zeros, we factor completely:
\[
P(x) = x^3 - 2x^2 - 4x + 8 = x^2(x - 2) - 4(x - 2) = (x^2 - 4)(x - 2)
\]
\[
= (x + 2)(x - 2)(x - 2)
\]
Thus the zeros are \( x = -2 \) and \( x = 2 \).

(b) The \( x \)-intercepts are \( x = -2 \) and \( x = 2 \). The \( y \)-intercept is \( P(0) = 8 \). The table gives additional values of \( P(x) \). Since \( P \) is of odd degree and its leading coefficient is positive, it has the following end behavior:
\[
y \to \infty \text{ as } x \to \infty \quad \text{and} \quad y \to -\infty \text{ as } x \to -\infty
\]
We plot the points in the table and connect them by a smooth curve to complete the graph.

\[
\begin{array}{c|c}
x & P(x) \\
-3 & -25 \\
-2 & 0 \\
-1 & 9 \\
0 & 8 \\
1 & 3 \\
2 & 0 \\
3 & 5 \\
\end{array}
\]

EXAMPLE: Let \( P(x) = x^3 + 3x^2 - 9x - 27 \).
(a) Find the zeros of \( P \).
(b) Sketch the graph of \( P \).

Answer:
(a) \( P(x) = (x + 3)^2(x - 3) \), so the zeros are \( x = -3, \ x = 3 \).

(b) End term behavior:
\[
y \to \infty \text{ as } x \to \infty \quad \text{and} \quad y \to -\infty \text{ as } x \to -\infty
\]
We use test point 0 to obtain the graph:
If $c$ is a zero of $P$ and the corresponding factor $x - c$ occurs exactly $m$ times in the factorization of $P$ then we say that $c$ is a zero of multiplicity $m$. One can show that the graph of $P$ crosses the $x$-axis at $c$ if the multiplicity $m$ is odd and does not cross the $x$-axis if $m$ is even. Moreover, it can be shown that near $x = c$ the graph has the same general shape as $y = A(x - c)^m$.

### Shape of the Graph Near a Zero of Multiplicity $m$

Suppose that $c$ is a zero of $P$ of multiplicity $m$. Then the shape of the graph of $P$ near $c$ is as follows.

<table>
<thead>
<tr>
<th>Multiplicity of $c$</th>
<th>Shape of the graph of $P$ near the $x$-intercept $c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$ odd, $m &gt; 1$</td>
<td>![Graph showing a shape near a zero with odd multiplicity.]</td>
</tr>
<tr>
<td>$m$ even, $m &gt; 1$</td>
<td>![Graph showing a shape near a zero with even multiplicity.]</td>
</tr>
</tbody>
</table>

**EXAMPLE:** Graph the polynomial $P(x) = x^4(x - 2)^3(x + 1)^2$.

Solution: The zeros of $P$ are $-1, 0,$ and $2$, with multiplicities $2, 4,$ and $3$, respectively.

- $0$ is a zero of multiplicity $4$.
- $2$ is a zero of multiplicity $3$.
- $-1$ is a zero of multiplicity $2$.

The zero 2 has odd multiplicity, so the graph crosses the $x$-axis at the $x$-intercept 2. But the zeros 0 and $-1$ have even multiplicity, so the graph does not cross the $x$-axis at the $x$-intercepts $0$ and $-1$.

Since $P$ is a polynomial of degree 9 and has positive leading coefficient, it has the following end behavior:

$y \to \infty$ as $x \to \infty$ and $y \to -\infty$ as $x \to -\infty$

With this information and a table of values, we sketch the graph.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$P(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1.3$</td>
<td>$-9.2$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$0$</td>
</tr>
<tr>
<td>$-0.5$</td>
<td>$-3.9$</td>
</tr>
<tr>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$1$</td>
<td>$-4$</td>
</tr>
<tr>
<td>$2$</td>
<td>$0$</td>
</tr>
<tr>
<td>$2.3$</td>
<td>$8.2$</td>
</tr>
</tbody>
</table>
Local Maxima and Minima of Polynomials

If the point \((a, f(a))\) is the highest point on the graph of \(f\) within some viewing rectangle, then \((a, f(a))\) is a **local maximum point** on the graph and if \((b, f(b))\) is the lowest point on the graph of \(f\) within some viewing rectangle, then \((b, f(b))\) is a **local minimum point**. The set of all local maximum and minimum points on the graph of a function is called its **local extrema**.

![Graph of a polynomial function with local maximum and minimum points labeled](image)

For a polynomial function the number of local extrema must be less than the degree, as the following principle indicates.

**Local Extrema of Polynomials**

If \(P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0\) is a polynomial of degree \(n\), then the graph of \(P\) has at most \(n - 1\) local extrema.

A polynomial of degree \(n\) may in fact have less than \(n - 1\) local extrema. For example, \(P(x) = x^3\) has no local extrema, even though it is of degree 3.

**EXAMPLE:** Determine how many local extrema each polynomial has.

(a) \(P_1(x) = x^4 + x^3 - 16x^2 - 4x + 48\)  
(b) \(P_2(x) = x^5 + 3x^4 - 5x^3 - 15x^2 + 4x - 15\)  
(c) \(P_3(x) = 7x^4 + 3x^2 - 10x\)

**Solution:**

(a) \(P_1\) has two local minimum points and one local maximum point, for a total of three local extrema.

(b) \(P_2\) has two local minimum points and two local maximum points, for a total of four local extrema.

(c) \(P_3\) has just one local extremum, a local minimum.
EXAMPLE: Determine how many local extrema each polynomial has.
(a) \( P_1(x) = x^3 - x \)  
(b) \( P_2(x) = x^4 - 8x^3 + 22x^2 - 24x + 5 \)

Solution:
(a) \( P_1 \) has one local minimum point and one local maximum point for a total of two local extrema.
(b) \( P_2 \) has two local minimum points and one local maximum point for a total of three local extrema.

EXAMPLE: Sketch the family of polynomials \( P(x) = x^4 - kx^2 + 3 \) for \( k = 0, 1, 2, 3, \) and 4. How does changing the value of \( k \) affect the graph?

Solution: The polynomials are graphed below. We see that increasing the value of \( k \) causes the two local minima to dip lower and lower.

EXAMPLE: Sketch the family of polynomials \( P(x) = x^3 - cx^2 \) for \( c = 0, 1, 2, \) and 3. How does changing the value of \( c \) affect the graph?
EXAMPLE: Sketch the family of polynomials $P(x) = x^3 - cx^2$ for $c = 0, 1, 2,$ and $3$. How does changing the value of $c$ affect the graph?

Solution: The polynomials

$$P_0(x) = x^3, \quad P_1(x) = x^3 - x^2, \quad P_2(x) = x^3 - 2x^2, \quad P_3(x) = x^3 - 3x^2$$

are graphed in the Figure below. We see that increasing the value of $c$ causes the graph to develop an increasingly deep “valley” to the right of the $y$-axis, creating a local maximum at the origin and a local minimum at a point in quadrant IV. This local minimum moves lower and farther to the right as $c$ increases. To see why this happens, factor $P(x) = x^2(x - c)$. The polynomial $P$ has zeros at 0 and $c$, and the larger $c$ gets, the farther to the right the minimum between 0 and $c$ will be.