

9.2

Graphing Simple Rational Functions

What you should learn

GOAL 1 Graph simple rational functions.

GOAL 2 Use the graph of a rational function to solve **real-life** problems, such as finding the average cost per calendar in **Example 3**.

Why you should learn it

▼ To solve **real-life** problems, such as finding the frequency of an approaching ambulance siren in **Exs. 47 and 48**.



GOAL 1 GRAPHING A SIMPLE RATIONAL FUNCTION

A **rational function** is a function of the form

$$f(x) = \frac{p(x)}{q(x)}$$

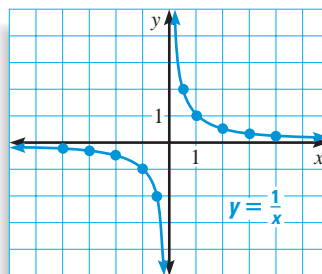
where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$. In this lesson you will learn to graph rational functions for which $p(x)$ and $q(x)$ are linear. For instance, consider the following rational function:

$$y = \frac{1}{x}$$

The graph of this function is called a **hyperbola** and is shown below. Notice the following properties.

- The x -axis is a horizontal asymptote.
- The y -axis is a vertical asymptote.
- The domain and range are all nonzero real numbers.
- The graph has two symmetrical parts called **branches**. For each point (x, y) on one branch, there is a corresponding point $(-x, -y)$ on the other branch.

x	y
-4	$-\frac{1}{4}$
-3	$-\frac{1}{3}$
-2	$-\frac{1}{2}$
-1	-1
$-\frac{1}{2}$	-2



x	y
4	$\frac{1}{4}$
3	$\frac{1}{3}$
2	$\frac{1}{2}$
1	1
$\frac{1}{2}$	2

ACTIVITY

Developing Concepts

Investigating Graphs of Rational Functions

- Graph each function.
 - $y = \frac{2}{x}$
 - $y = \frac{3}{x}$
 - $y = \frac{-1}{x}$
 - $y = \frac{-2}{x}$
- Use the graphs to describe how the sign of a affects the graph of $y = \frac{a}{x}$.
- Use the graphs to describe how $|a|$ affects the graph of $y = \frac{a}{x}$.

All rational functions of the form $y = \frac{a}{x - h} + k$ have graphs that are hyperbolas with asymptotes at $x = h$ and $y = k$. To draw the graph, plot a couple of points on each side of the vertical asymptote. Then draw the two branches of the hyperbola that approach the asymptotes and pass through the plotted points.

EXAMPLE 1 Graphing a Rational Function

Graph $y = \frac{-2}{x + 3} - 1$. State the domain and range.

STUDENT HELP

Look Back
For help with asymptotes, see p. 465.

SOLUTION

Draw the asymptotes $x = -3$ and $y = -1$.

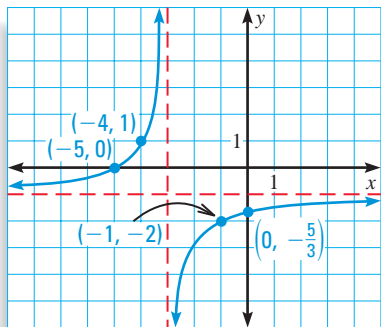
Plot two points to the left of the vertical asymptote, such as $(-4, 1)$ and $(-5, 0)$, and two points to the right, such as $(-1, -2)$ and $(0, -\frac{5}{3})$.

Use the asymptotes and plotted points to draw the branches of the hyperbola.

The domain is all real numbers except -3 , and the range is all real numbers except -1 .

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All rational functions of the form $y = \frac{ax + b}{cx + d}$ also have graphs that are hyperbolas. The vertical asymptote occurs at the x -value that makes the denominator zero. The horizontal asymptote is the line $y = \frac{a}{c}$.



EXAMPLE 2 Graphing a Rational Function

Graph $y = \frac{x + 1}{2x - 4}$. State the domain and range.

STUDENT HELP

INTERNET
HOMEWORK HELP
Visit our Web site
www.mcdougallittell.com
for extra examples.

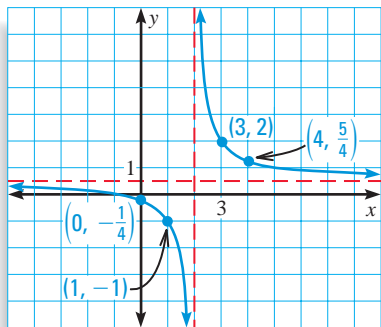
SOLUTION

Draw the asymptotes. Solve $2x - 4 = 0$ for x to find the vertical asymptote $x = 2$. The horizontal asymptote is the line $y = \frac{a}{c} = \frac{1}{2}$.

Plot two points to the left of the vertical asymptote, such as $(0, -\frac{1}{4})$ and $(1, -1)$, and two points to the right, such as $(3, 2)$ and $(4, \frac{5}{4})$.

Use the asymptotes and plotted points to draw the branches of the hyperbola.

The domain is all real numbers except 2 , and the range is all real numbers except $\frac{1}{2}$.



GOAL 2

USING RATIONAL FUNCTIONS IN REAL LIFE



EXAMPLE 3

Writing a Rational Model

For a fundraising project, your math club is publishing a fractal art calendar. The cost of the digital images and the permission to use them is \$850. In addition to these “one-time” charges, the *unit cost* of printing each calendar is \$3.25.

- Write a model that gives the average cost per calendar as a function of the number of calendars printed.
- Graph the model and use the graph to estimate the number of calendars you need to print before the average cost drops to \$5 per calendar.
- Describe what happens to the average cost as the number of calendars printed increases.



SOLUTION

- The average cost is the total cost of making the calendars divided by the number of calendars printed.



VERBAL MODEL

LABELS

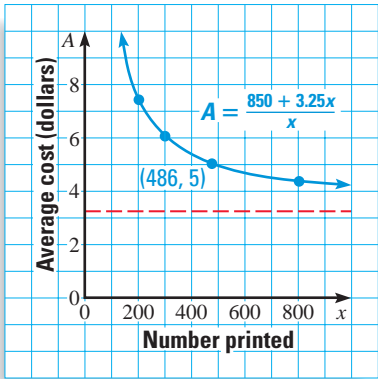
ALGEBRAIC MODEL

Average cost = $\frac{\text{One-time charges} + \text{Unit cost} \cdot \text{Number printed}}{\text{Number printed}}$

Average cost = A (dollars per calendar)
One-time charges = 850 (dollars)
Unit cost = 3.25 (dollars per calendar)
Number printed = x (calendars)

$A = \frac{850 + 3.25x}{x}$

- The graph of the model is shown at the right. The A -axis is the vertical asymptote and the line $A = 3.25$ is the horizontal asymptote. The domain is $x > 0$ and the range is $A > 3.25$. When $A = 5$ the value of x is about 500. So, you need to print about 500 calendars before the average cost drops to \$5 per calendar.
- As the number of calendars printed increases, the average cost per calendar gets closer and closer to \$3.25. For instance, when $x = 5000$ the average cost is \$3.42, and when $x = 10,000$ the average cost is \$3.34.



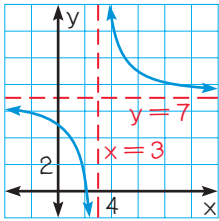
GUIDED PRACTICE

Vocabulary Check

1. Complete this statement: The graph of a function of the form $y = \frac{a}{x-h} + k$ is called a(n) .

Concept Check

2. **ERROR ANALYSIS** Explain why the graph shown is not the graph of $y = \frac{6}{x+3} + 7$.



3. If the graph of a rational function is a hyperbola with the x -axis and the y -axis as asymptotes, what is the domain of the function? What is the range?

Skill Check

Identify the horizontal and vertical asymptotes of the graph of the function.

4. $y = \frac{2}{x-3} + 4$

5. $y = \frac{2x+3}{x+4}$

6. $y = \frac{x-3}{x+3}$

7. $y = \frac{x+5}{2x-4}$

8. $y = \frac{3}{x+8} - 10$

9. $y = \frac{-4}{x-6} - 5$

10. **CALENDAR FUNDRAISER** Look back at Example 3 on page 542. Suppose you decide to generate your own fractals on a computer to save money. The cost for the software (a “one-time” cost) is \$125. Write a model that gives the average cost per calendar as a function of the number of calendars printed. Graph the model and use the graph to estimate the number of calendars you need to print before the average cost drops to \$5 per calendar.

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice
to help you master
skills is on p. 952.

IDENTIFYING ASYMPTOTES Identify the horizontal and vertical asymptotes of the graph of the function. Then state the domain and range.

11. $y = \frac{3}{x} + 2$

12. $y = \frac{4}{x-3} + 2$

13. $y = \frac{-2}{x+3} - 2$

14. $y = \frac{x+2}{x-3}$

15. $y = \frac{2x+2}{3x+1}$

16. $y = \frac{-3x+2}{-4x-5}$

17. $y = \frac{-22}{x+43} - 17$

18. $y = \frac{34x-2}{16x+4}$

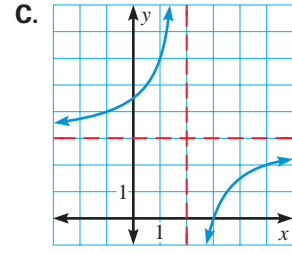
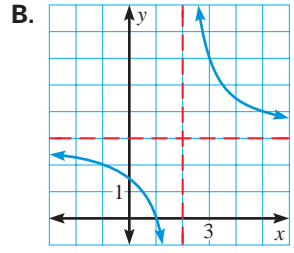
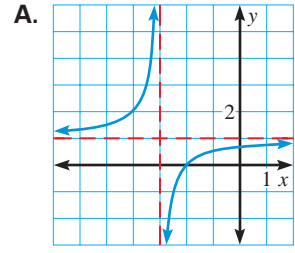
19. $y = \frac{4}{x-6} + 19$

MATCHING GRAPHS Match the function with its graph.

20. $y = \frac{3}{x-2} + 3$

21. $y = \frac{-3}{x-2} + 3$

22. $y = \frac{x+2}{x+3}$



STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 11–31
Example 2: Exs. 11–22,
32–40
Example 3: Exs. 42–48

GRAPHING FUNCTIONS Graph the function. State the domain and range.

23. $y = \frac{4}{x}$

24. $y = \frac{3}{x-3} + 1$

25. $y = \frac{-4}{x+5} - 8$

26. $y = \frac{1}{x-7} + 3$

27. $y = \frac{6}{x+2} - 6$

28. $y = \frac{5}{x} + 4$

29. $y = \frac{1}{4x+12} - 2$

30. $y = \frac{3}{2x}$

31. $y = \frac{4}{3x-6} + 5$

GRAPHING FUNCTIONS Graph the function. State the domain and range.

32. $y = \frac{x+2}{x+3}$

33. $y = \frac{x}{4x+3}$

34. $y = \frac{x-7}{3x-8}$

35. $y = \frac{9x+1}{3x-2}$

36. $y = \frac{-3x+10}{4x-12}$

37. $y = \frac{5x+2}{4x}$

38. $y = \frac{3x}{2x-4}$

39. $y = \frac{7x}{-x-15}$

40. $y = \frac{-14x-4}{2x-1}$

41. **CRITICAL THINKING** Write a rational function that has the vertical asymptote $x = -4$ and the horizontal asymptote $y = 3$.

RACQUETBALL In Exercises 42 and 43, use the following information.

You've paid \$120 for a membership to a racquetball club. Court time is \$5 per hour.

42. Write a model that represents your average cost per hour of court time as a function of the number of hours played. Graph the model. What is an equation of the horizontal asymptote and what does the asymptote represent?
43. Suppose that you can play racquetball at the YMCA for \$9 per hour without being a member. How many hours would you have to play at the racquetball club before your average cost per hour of court time is less than \$9?
44. **LIGHTNING** Air temperature affects how long it takes sound to travel a given distance. The time it takes for sound to travel one kilometer can be modeled by

$$t = \frac{1000}{0.6T + 331}$$

where t is the time (in seconds) and T is the temperature (in degrees Celsius). You are 1 kilometer from a lightning strike and it takes you exactly 3 seconds to hear the sound of thunder. Use a graph to find the approximate air temperature. (*Hint:* Use tick marks that are 0.1 unit apart on the t -axis.)

ECONOMICS In Exercises 45 and 46, use the following information.

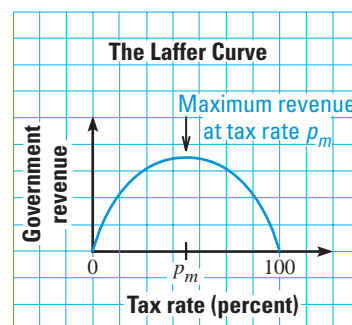
Economist Arthur Laffer argues that beyond a certain percent p_m , increased taxes will produce less government revenue. His theory is illustrated in the graph below.

45. Using Laffer's theory, an economist models the revenue generated by one kind of tax by

$$R = \frac{80p - 8000}{p - 110}$$

where R is the government revenue (in tens of millions of dollars) and p is the percent tax rate ($55 \leq p \leq 100$). Graph the model.

46. Use your graph from Exercise 45 to find the tax rate that yields \$600 million of revenue.

**FOCUS ON APPLICATIONS****REAL LIFE LIGHTNING**

The equation given in Ex. 44 uses the fact that at 0°C , sound travels at 331 meters per second. Since light travels at about 300,000 kilometers per second, a flash of lightning is seen before it is heard.

**DOPPLER EFFECT** In Exercises 47 and 48, use the following information.

When the source of a sound is moving relative to a stationary listener, the frequency f_l (in hertz) heard by the listener is different from the frequency f_s (in hertz) of the sound at its source. An equation for the frequency heard by the listener is

$$f_l = \frac{740f_s}{740 - r}$$

where r is the speed (in miles per hour) of the sound source relative to the listener.

47. The sound of an ambulance siren has a frequency of about 2000 hertz. You are standing on the sidewalk as an ambulance approaches with its siren on. Write the frequency that you hear as a function of the ambulance's speed.
48. Graph the function from Exercise 47 for $0 \leq r \leq 60$. What happens to the frequency you hear as the value of r increases?

49. **Writing** In what line(s) is the graph of $y = \frac{1}{x}$ symmetric? What does this symmetry tell you about the inverse of the function $f(x) = \frac{1}{x}$?

50. **MULTIPLE CHOICE** What are the asymptotes of the graph of $y = \frac{-73}{x - 141} + 27$?

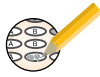
- (A) $x = 141, y = 27$ (B) $x = -141, y = 27$ (C) $x = -73, y = 27$
 (D) $x = -73, y = 141$ (E) None of these

51. **MULTIPLE CHOICE** Which of the following is a function whose domain and range are all *nonzero* real numbers?

- (A) $f(x) = \frac{x}{2x + 1}$ (B) $f(x) = \frac{2x - 1}{3x - 2}$ (C) $f(x) = \frac{1}{x} + 1$
 (D) $f(x) = \frac{x - 2}{x}$ (E) None of these

★ Challenge

52. **EQUIVALENT FORMS** Show algebraically that the function $f(x) = \frac{3}{x - 5} + 10$ and the function $g(x) = \frac{10x - 47}{x - 5}$ are equivalent.

Test Preparation**MIXED REVIEW****GRAPHING POLYNOMIALS** Graph the polynomial function. (Review 6.2 for 9.3)

53. $f(x) = 3x^5$ 54. $f(x) = 4 - 2x^3$ 55. $f(x) = x^6 - 1$
 56. $f(x) = 4x^4 + 1$ 57. $f(x) = 6x^7$ 58. $f(x) = x^3 - 5$

FACTORING Factor the polynomial. (Review 6.4 for 9.3)

59. $8x^3 - 125$ 60. $3x^3 + 81$ 61. $x^3 + 3x^2 + 3x + 9$
 62. $5x^3 + 10x^2 + x + 2$ 63. $81x^4 - 1$ 64. $4x^4 - 4x^2 - 120$

SIMPLIFYING EXPRESSIONS Simplify the expression. (Review 8.3)

65. $\frac{e^x}{e^x}$ 66. $7e^{-5}e^8$ 67. e^xe^{4x+1}
 68. $\frac{6e^x}{e^x}$ 69. $e^4e^{2x}e^{-3x}$ 70. e^3e^{-5}