8.4 Logarithmic Functions

What *you should learn*

GOAL 1 Evaluate logarithmic functions.

GOAL 2 Graph logarithmic functions, as applied in **Example 8**.

Why *you should learn it*

▼ To model **real-life** situations, such as the slope of a beach in **Example 4**.

REAL LIFE

GOAL 1 EVALUATING LOGARITHMIC FUNCTIONS

You know that $2^2 = 4$ and $2^3 = 8$. However, for what value of x does $2^x = 6$? Because $2^2 < 6 < 2^3$, you would expect x to be between 2 and 3. To find the exact x-value, mathematicians defined *logarithms.* In terms of a logarithm, $x = \log_2 6 \approx 2.585$. (In the next lesson you will see how this x-value is obtained.)

DEFINITION OF LOGARITHM WITH BASE *b*

Let b and y be positive numbers, $b \neq 1$. The **logarithm of *y* with base *b*** is denoted by $\log_b y$ and is defined as follows:

$$\log_b y = x \text{ if and only if } b^x = y$$

The expression $\log_b y$ is read as "log base b of y."

This definition tells you that the equations $\log_b y = x$ and $b^x = y$ are equivalent. The first is in *logarithmic form* and the second is in *exponential form.* Given an equation in one of these forms, you can always rewrite it in the other form.

EXAMPLE 1 *Rewriting Logarithmic Equations*

	LOGARITHMIC FORM	EXPONENTIAL FORM
a.	$\log_2 32 = 5$	$2^5 = 32$
b.	$\log_5 1 = 0$	$5^0 = 1$
c.	$\log_{10} 10 = 1$	$10^1 = 10$
d.	$\log_{10} 0.1 = -1$	$10^{-1} = 0.1$
e.	$\log_{1/2} 2 = -1$	$\left(\frac{1}{2}\right)^{-1} = 2$

Parts (b) and (c) of Example 1 illustrate two special logarithm values that you should learn to recognize.

SPECIAL LOGARITHM VALUES

Let b be a positive real number such that $b \neq 1$.

LOGARITHM OF 1 $\quad \log_b 1 = 0$ because $b^0 = 1$.

LOGARITHM OF BASE *b* $\quad \log_b b = 1$ because $b^1 = b$.

EXAMPLE 2 *Evaluating Logarithmic Expressions*

STUDENT HELP

INTERNET **HOMEWORK HELP** Visit our Web site www.mcdougallittell.com for extra examples.

Evaluate the expression.

a. $\log_3 81$ **b.** $\log_5 0.04$ **c.** $\log_{1/2} 8$ **d.** $\log_9 3$

SOLUTION

To help you find the value of $\log_b y$, ask yourself what power of b gives you y.

a. 3 to what power gives 81?

$3^4 = 81$, so $\log_3 81 = 4$.

b. 5 to what power gives 0.04?

$5^{-2} = 0.04$, so $\log_5 0.04 = -2$.

c. $\frac{1}{2}$ to what power gives 8?

$\left(\frac{1}{2}\right)^{-3} = 8$, so $\log_{1/2} 8 = -3$.

d. 9 to what power gives 3?

$9^{1/2} = 3$, so $\log_9 3 = \frac{1}{2}$.

. . . .

The logarithm with base 10 is called the **common logarithm**. It is denoted by $\log_{10}$ or simply by log. The logarithm with base e is called the **natural logarithm**. It can be denoted by $\log_e$, but it is more often denoted by ln.

COMMON LOGARITHM	NATURAL LOGARITHM
$\log_{10} x = \log x$	$\log_e x = \ln x$

Most calculators have keys for evaluating common and natural logarithms.

EXAMPLE 3 *Evaluating Common and Natural Logarithms*

EXPRESSION	KEYSTROKES	DISPLAY
a. log 5	LOG 5 ENTER	0.698970
b. ln 0.1	LN .1 ENTER	-2.302585

FOCUS ON APPLICATIONS

Sand particle	Diameter (mm)
Pebble	4
Granule	2
Very coarse sand	1
Coarse sand	0.5
Medium sand	0.25
Fine sand	0.125
Very fine sand	0.0625

REAL LIFE **SAND** The table gives the diameters of different types of sand. Notice that the diameter of a pebble is about 64 times larger than the diameter of very fine sand.

EXAMPLE 4 *Evaluating a Logarithmic Function*

SCIENCE CONNECTION The slope s of a beach is related to the average diameter d (in millimeters) of the sand particles on the beach by this equation:

$$s = 0.159 + 0.118 \log d$$

Find the slope of a beach if the average diameter of the sand particles is 0.25 millimeter.

SOLUTION

If $d = 0.25$, then the slope of the beach is:

$s = 0.159 + 0.118 \log 0.25$ **Substitute 0.25 for *d*.**

$\approx 0.159 + 0.118(-0.602)$ **Use a calculator.**

≈ 0.09 **Simplify.**

▶ The slope of the beach is about 0.09. This is a gentle slope that indicates a rise of only 9 meters for a run of 100 meters.

GOAL 2 GRAPHING LOGARITHMIC FUNCTIONS

By the definition of a logarithm, it follows that the logarithmic function $g(x) = \log_b x$ is the inverse of the exponential function $f(x) = b^x$. This means that:

$$g(f(x)) = \log_b b^x = x \qquad \text{and} \qquad f(g(x)) = b^{\log_b x} = x$$

In other words, exponential functions and logarithmic functions "undo" each other.

STUDENT HELP

Look Back
For help with inverses, see p. 422.

EXAMPLE 5 *Using Inverse Properties*

Simplify the expression.

a. $10^{\log 2}$ **b.** $\log_3 9^x$

SOLUTION

a. $10^{\log 2} = 2$ **b.** $\log_3 9^x = \log_3 (3^2)^x = \log_3 3^{2x} = 2x$

EXAMPLE 6 *Finding Inverses*

Find the inverse of the function.

a. $y = \log_3 x$ **b.** $y = \ln (x + 1)$

SOLUTION

a. From the definition of logarithm, the inverse of $y = \log_3 x$ is $y = 3^x$.

b.
$y = \ln (x + 1)$ **Write original function.**

$x = \ln (y + 1)$ **Switch *x* and *y*.**

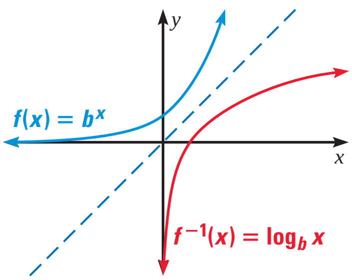
$e^x = y + 1$ **Write in exponential form.**

$e^x - 1 = y$ **Solve for *y*.**

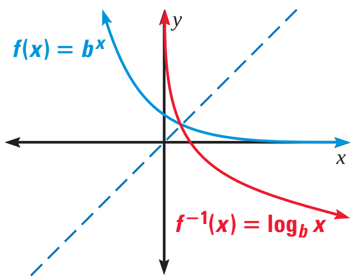
▶ The inverse of $y = \ln (x + 1)$ is $y = e^x - 1$.

The inverse relationship between exponential and logarithmic functions is also useful for graphing logarithmic functions. Recall from Lesson 7.4 that the graph of f^{-1} is the reflection of the graph of f in the line $y = x$.

Graphs of f and f^{-1} for $b > 1$



Graphs of f and f^{-1} for $0 < b < 1$



CONCEPT SUMMARY — GRAPHS OF LOGARITHMIC FUNCTIONS

The graph of $y = \log_b (x - h) + k$ has the following characteristics:

- The line $x = h$ is a vertical asymptote.
- The domain is $x > h$, and the range is all real numbers.
- If $b > 1$, the graph moves up to the right. If $0 < b < 1$, the graph moves down to the right.

EXAMPLE 7 *Graphing Logarithmic Functions*

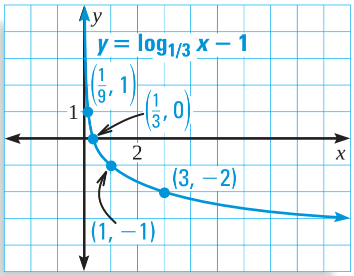
Graph the function. State the domain and range.

a. $y = \log_{1/3} x - 1$

b. $y = \log_5 (x + 2)$

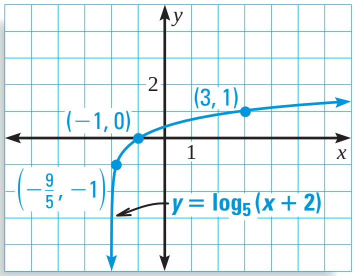
SOLUTION

a. Plot several convenient points, such as $\left(\frac{1}{3}, 0\right)$ and $(3, -2)$. The vertical line $x = 0$ is an asymptote. From left to right, draw a curve that starts just to the right of the y-axis and moves down.



The domain is $x > 0$, and the range is all real numbers.

b. Plot several convenient points, such as $(-1, 0)$ and $(3, 1)$. The vertical line $x = -2$ is an asymptote. From left to right, draw a curve that starts just to the right of the line $x = -2$ and moves up.



The domain is $x > -2$, and the range is all real numbers.

FOCUS ON APPLICATIONS

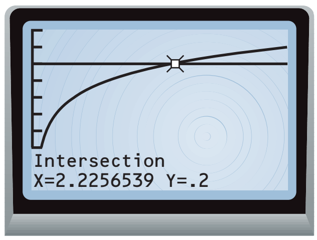
REAL LIFE **SAND** A beach with very fine sand makes an angle of about 1° with the horizontal while a beach with pebbles makes an angle of about 17°.

EXAMPLE 8 *Using the Graph of a Logarithmic Function*

SCIENCE CONNECTION Graph the model from Example 4, $s = 0.159 + 0.118 \log d$. Then use the graph to estimate the average diameter of the sand particles for a beach whose slope is 0.2.

SOLUTION

You can use a graphing calculator to graph the model. Then, using the *Intersect* feature, you can determine that $s = 0.2$ when $d \approx 2.23$, as shown at the right. So, the average diameter of the sand particles is about 2.23 millimeters.



GUIDED PRACTICE

Vocabulary Check ✓

1. Complete this statement: The logarithm with base 10 is called the _?_.

Concept Check ✓

2. Explain why the expressions $\log_3(-1)$ and $\log_1 1$ are not defined.
3. Explain the meaning of $\log_b y$.
4. **ERROR ANALYSIS** To simplify $\log_2 25$, a student reasoned as shown. Describe the error that the student made.

$$5^2 = 25$$
$$\text{so } \log_2 25 = 5$$

Skill Check ✓

Rewrite the equation in exponential form.

5. $\log_3 9 = 2$
6. $\log_5 5 = 1$
7. $\log_{1/2} 4 = -2$
8. $\log_{19} 1 = 0$

Evaluate the expression.

9. $\log_2 64$
10. $\log_{25} 5$
11. $\log_6 1$
12. $10^{\log 4}$

Graph the function. State the domain and range.

13. $y = \log_2(x + 1) - 3$
14. $y = \log_{1/2}(x - 2) + 1$
15. **SLOPE OF A BEACH** Using the model from Example 4 and a graphing calculator, find the average diameter of the sand particles for a beach whose slope is 0.1.

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 951.

REWRITING IN EXPONENTIAL FORM **Rewrite the equation in exponential form.**

16. $\log_4 1024 = 5$
17. $\log_5 \frac{1}{5} = -1$
18. $\log_{36} \frac{1}{6} = -\frac{1}{2}$
19. $\log_8 512 = 3$
20. $\log_{12} 144 = 2$
21. $\log_{14} 196 = 2$
22. $\log_8 4096 = 4$
23. $\log_{105} 11{,}025 = 2$

EVALUATING EXPRESSIONS **Evaluate the expression without using a calculator.**

24. $\log_5 125$
25. $\log_7 343$
26. $\log_8 1$
27. $\log_{12} 12$
28. $\log_6 36$
29. $\log_4 16$
30. $\log_9 729$
31. $\log_7 2401$
32. $\log_{1/4} \frac{1}{4}$
33. $\log_4 4^{-0.38}$
34. $\log_4 \frac{1}{2}$
35. $\log_{1/5} 25$

EVALUATING LOGARITHMS **Use a calculator to evaluate the expression. Round the result to three decimal places.**

36. $\log 8$
37. $\ln 10$
38. $\log \sqrt{2}$
39. $\log 3.724$
40. $\log 2.54$
41. $\log 0.3$
42. $\log 4.05$
43. $\log 3.5$
44. $\ln 4.6$
45. $\ln 150$
46. $\ln 6.9$
47. $\ln 22.5$

USING INVERSES **Simplify the expression.**

48. $5^{\log_5 x}$
49. $\log_2 2^x$
50. $9^{\log_9 x}$
51. $35^{\log_{35} x}$
52. $\log_4 16^x$
53. $7^{\log_7 x}$
54. $\log 100^x$
55. $\log_{20} 8000^x$

STUDENT HELP

HOMEWORK HELP

Example 1: Exs. 16–23
Example 2: Exs. 24–35
Example 3: Exs. 36–47
Example 4: Exs. 77–79
Example 5: Exs. 48–55
Example 6: Exs. 56–64
Example 7: Exs. 65–76
Example 8: Exs. 80, 81

FINDING INVERSES **Find the inverse of the function.**

56. $y = \log_9 x$ **57.** $y = \log_{1/4} x$ **58.** $y = \log_5 x$

59. $y = \log_{1/2} x$ **60.** $y = \log_7 49^x$ **61.** $y = \ln 6x$

62. $y = \ln (x - 1)$ **63.** $y = \ln (x + 2)$ **64.** $y = \ln (x - 2)$

GRAPHING FUNCTIONS **Graph the function. State the domain and range.**

65. $y = \log_5 x$ **66.** $y = \ln x + 3$ **67.** $y = \log_2 x + 1$

68. $y = \ln x - 1$ **69.** $y = \log_8 x - 2$ **70.** $y = \ln (x + 1)$

71. $y = \log (x - 2)$ **72.** $y = \ln (x - 2)$ **73.** $y = \log_5 (x + 4)$

74. $y = \log_{1/2} x - 1$ **75.** $y = \log_{1/4} x - 3$ **76.** $y = \ln x + 5$

77. **SCIENCE CONNECTION** The pH of a solution is given by the formula

$$\text{pH} = -\log [H^+]$$

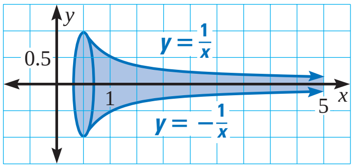
where $[H^+]$ is the solution's hydrogen ion concentration (in moles per liter). Find the pH of the solution.

a. lemon juice: $[H^+] = 1 \times 10^{-2.4}$ moles per liter

b. vinegar: $[H^+] = 1 \times 10^{-3}$ moles per liter

c. orange juice: $[H^+] = 1 \times 10^{-3.5}$ moles per liter

78. **GEOMETRY CONNECTION** Part of the three-dimensional mathematical figure called the horn of Gabriel is shown. The area of the cross section (in the coordinate plane) of the horn is given by:



$$A = \frac{2}{\log e}$$

Approximate this area to three decimal places.

SEISMOLOGY **In Exercises 79 and 80, use the following information.**
The Richter scale is used for measuring the magnitude of an earthquake. The Richter magnitude R is given by the model

$$R = 0.67 \log (0.37E) + 1.46$$

where E is the energy (in kilowatt-hours) released by the earthquake.

79. Suppose an earthquake releases 15,500,000,000 kilowatt-hours of energy. What is the earthquake's magnitude? (Use a calculator.)

80. How many kilowatt-hours of energy would the earthquake in Exercise 79 have to release in order to increase its magnitude by one-half of a unit on the Richter scale? Use a graph to solve the problem.

81. **TORNADOES** Most tornadoes last less than an hour and travel less than 20 miles. The wind speed s (in miles per hour) near the center of a tornado is related to the distance d (in miles) the tornado travels by this model:

$$s = 93 \log d + 65$$

On March 18, 1925, a tornado whose wind speed was about 280 miles per hour struck the Midwest. Use a graph to estimate how far the tornado traveled.

FOCUS ON APPLICATIONS

REAL LIFE **TORNADOES** form along a boundary between warm humid air from the Gulf of Mexico and cool dry air from the north. When thunderstorm clouds develop along this boundary, the result is violent weather which can produce a tornado.

INTERNET **APPLICATION LINK**
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Test Preparation

QUANTITATIVE COMPARISON **In Exercises 82–87 choose the statement that is true about the given quantities.**

Ⓐ The quantity in column A is greater.

Ⓑ The quantity in column B is greater.

Ⓒ The two quantities are equal.

Ⓓ The relationship cannot be determined from the given information.

	Column A	Column B
82.	$\log_9 9^{2/3}$	$\log 100$
83.	$\log_{16} 1$	0
84.	$\log_4 16$	$\log_8 64$
85.	$f(8)$ if $f(x) = \log_2 x$	4
86.	$f(-1)$ if $f(x) = \log_5 5^x$	-1
87.	$f\left(\frac{1}{2}\right)$ if $f(x) = \log_3 9^x$	$\log_3 81$

★ **Challenge**

EVALUATING EXPRESSIONS **Evaluate the expression. (*Hint:* Each expression has the form $\log_b x$. Rewrite the base b and the x-value as powers of the same number.)**

88. $\log_{16} 8$ **89.** $\log_{16} 64$ **90.** $\log_9 27$ **91.** $\log_4 512$

EXTRA CHALLENGE
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92. CRITICAL THINKING What pattern do you recognize in your answers to Exercises 88–91?

MIXED REVIEW

EVALUATING NUMERICAL EXPRESSIONS **Evaluate the numerical expression.** (Review 6.1 for 8.5)

93. $5^2 \cdot 5^3$ **94.** $(3^{-4})^2$ **95.** $7^0 \cdot 7^3 \cdot 7^{-2}$ **96.** $\left(\frac{3}{7}\right)^{-2}$

97. $\frac{6^3}{6^4}$ **98.** $\left(\frac{3}{8}\right)^{-3}$ **99.** $(-2^3)^2$ **100.** $\left(\frac{4}{5}\right)^3$

101. $\left(\frac{1}{2}\right)^{-4}$ **102.** $(-3^2)^{-1}$ **103.** $\frac{2^5}{2^9}$ **104.** $\left(\frac{7}{9}\right)^{-2}$

USING LONG DIVISION **Divide using long division.** (Review 6.5)

105. $(2x^2 + x - 1) \div (x + 4)$ **106.** $(x^2 - 5x + 4) \div (x - 1)$

107. $(4x^3 + 3x^2 + 2x - 3) \div (x^2 + 2)$ **108.** $(6x^3 - 8x^2 + 7) \div (x - 3)$

FINDING A CUBIC MODEL **Write a cubic function whose graph passes through the given points.** (Review 6.9)

109. $(2, 0), (-3, 0), (0, 0), (3, -3)$ **110.** $(3, 0), (2, 0), (-3, 0), (0, -1)$

111. $(4, 0), (6, 0), (-4, 0), (1, 1)$ **112.** $(-2, 0), (-3, 0), (3, 0), (0, 2)$