7.4 Inverse Functions

***What** you should learn*

GOAL 1 Find inverses of linear functions.

GOAL 2 Find inverses of nonlinear functions, as applied in **Example 6**.

***Why** you should learn it*

▼ To solve **real-life** problems, such as finding your bowling average in **Ex. 59**.

REAL LIFE

GOAL 1 FINDING INVERSES OF LINEAR FUNCTIONS

In Lesson 2.1 you learned that a *relation* is a mapping of input values onto output values. An **inverse relation** maps the output values back to their original input values. This means that the domain of the inverse relation is the range of the original relation and that the range of the inverse relation is the domain of the original relation.

Original relation

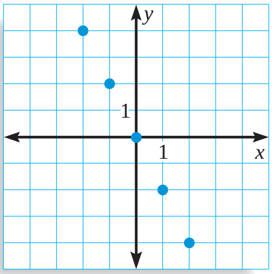
x	-2	-1	0	1	2
y	4	2	0	-2	-4

Inverse relation

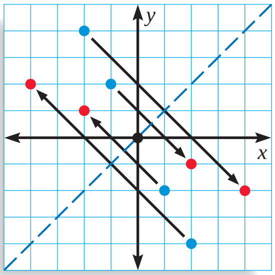
x	4	2	0	-2	-4
y	-2	-1	0	1	2

The graph of an inverse relation is the *reflection* of the graph of the original relation. The line of reflection is $y = x$.

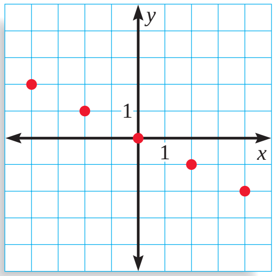
Graph of original relation



Reflection in $y = x$



Graph of inverse relation



To find the inverse of a relation that is given by an equation in x and y, switch the roles of x and y and solve for y (if possible).

EXAMPLE 1 *Finding an Inverse Relation*

Find an equation for the inverse of the relation $y = 2x - 4$.

SOLUTION

$y = 2x - 4$	Write original relation.
$x = 2y - 4$	Switch x and y.
$x + 4 = 2y$	Add 4 to each side.
$\frac{1}{2}x + 2 = y$	Divide each side by 2.

▶ The inverse relation is $y = \frac{1}{2}x + 2$.

STUDENT HELP

Look Back
For help with solving equations for y, see p. 26.

In Example 1 both the original relation and the inverse relation happen to be functions. In such cases the two functions are called **inverse functions**.

STUDENT HELP

Study Tip
The notation for an inverse function, f^{-1}, looks like a negative exponent, but it should not be interpreted that way. In other words, $f^{-1}(x) \neq (f(x))^{-1} = \frac{1}{f(x)}$.

INVERSE FUNCTIONS

Functions f and g are inverses of each other provided:

$$f(g(x)) = x \qquad \text{and} \qquad g(f(x)) = x$$

The function g is denoted by f^{-1}, read as "f inverse."

Given any function, you can always find its inverse relation by switching x and y. For a linear function $f(x) = mx + b$ where $m \neq 0$, the inverse is itself a linear function.

EXAMPLE 2 *Verifying Inverse Functions*

Verify that $f(x) = 2x - 4$ and $f^{-1}(x) = \frac{1}{2}x + 2$ are inverses.

SOLUTION Show that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

$$\begin{aligned} f(f^{-1}(x)) &= f\left(\frac{1}{2}x + 2\right) \\ &= 2\left(\frac{1}{2}x + 2\right) - 4 \\ &= x + 4 - 4 \\ &= x\ \checkmark \end{aligned} \qquad \begin{aligned} f^{-1}(f(x)) &= f^{-1}(2x - 4) \\ &= \frac{1}{2}(2x - 4) + 2 \\ &= x - 2 + 2 \\ &= x\ \checkmark \end{aligned}$$

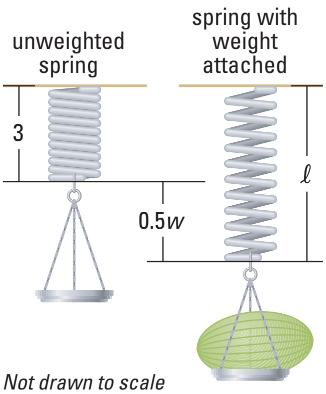
REAL LIFE Science

EXAMPLE 3 *Writing an Inverse Model*

When calibrating a spring scale, you need to know how far the spring stretches based on given weights. Hooke's law states that the length a spring stretches is proportional to the weight attached to the spring. A model for one scale is $\ell = 0.5w + 3$ where ℓ is the total length (in inches) of the spring and w is the weight (in pounds) of the object.

a. Find the inverse model for the scale.

b. If you place a melon on the scale and the spring stretches to a total length of 5.5 inches, how much does the melon weigh?



Not drawn to scale

SOLUTION

a.

$\ell = 0.5w + 3$	**Write original model.**
$\ell - 3 = 0.5w$	**Subtract 3 from each side.**
$\frac{\ell - 3}{0.5} = w$	**Divide each side by 0.5.**
$2\ell - 6 = w$	**Simplify.**

b. To find the weight of the melon, substitute 5.5 for ℓ.

$$w = 2\ell - 6 = 2(5.5) - 6 = 11 - 6 = 5$$

▶ The melon weighs 5 pounds.

STUDENT HELP

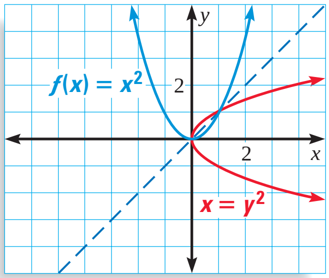
Study Tip
Notice that you do not switch the variables when you are finding inverses for models. This would be confusing because the letters are chosen to remind you of the real-life quantities they represent.

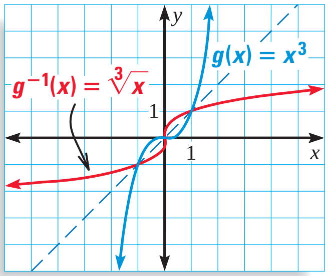
GOAL 2 FINDING INVERSES OF NONLINEAR FUNCTIONS

The graphs of the power functions $f(x) = x^2$ and $g(x) = x^3$ are shown below along with their reflections in the line $y = x$. Notice that the inverse of $g(x) = x^3$ is a function, but that the inverse of $f(x) = x^2$ is *not* a function.

STUDENT HELP

Look Back
For help with recognizing when a relationship is a function, see p. 70.





If the domain of $f(x) = x^2$ is *restricted,* say to only nonnegative real numbers, then the inverse of f *is* a function.

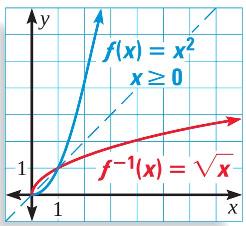
EXAMPLE 4 *Finding an Inverse Power Function*

Find the inverse of the function $f(x) = x^2, x \geq 0$.

SOLUTION

$f(x) = x^2$	**Write original function.**
$y = x^2$	**Replace $f(x)$ with y.**
$x = y^2$	**Switch x and y.**
$\pm\sqrt{x} = y$	**Take square roots of each side.**

Because the domain of f is restricted to nonnegative values, the inverse function is $f^{-1}(x) = \sqrt{x}$. (You would choose $f^{-1}(x) = -\sqrt{x}$ if the domain had been restricted to $x \leq 0$.)



✓**CHECK** To check your work, graph f and f^{-1} as shown. Note that the graph of $f^{-1}(x) = \sqrt{x}$ is the reflection of the graph of $f(x) = x^2, x \geq 0$ in the line $y = x$.

In the graphs at the top of the page, notice that the graph of $f(x) = x^2$ can be intersected twice with a horizontal line and that its inverse is *not* a function. On the other hand, the graph of $g(x) = x^3$ cannot be intersected twice with a horizontal line and its inverse *is* a function. This observation suggests the *horizontal line test.*

HORIZONTAL LINE TEST

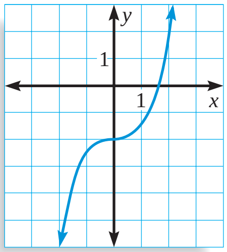
If no horizontal line intersects the graph of a function f more than once, then the inverse of f is itself a function.

EXAMPLE 5 *Finding an Inverse Function*

Consider the function $f(x) = \frac{1}{2}x^3 - 2$. Determine whether the inverse of f is a function. Then find the inverse.

SOLUTION

Begin by graphing the function and noticing that no horizontal line intersects the graph more than once. This tells you that the inverse of f is itself a function. To find an equation for f^{-1}, complete the following steps.



$$f(x) = \frac{1}{2}x^3 - 2 \qquad \text{Write original function.}$$

$$y = \frac{1}{2}x^3 - 2 \qquad \text{Replace } f(x) \text{ with } y.$$

$$x = \frac{1}{2}y^3 - 2 \qquad \text{Switch } x \text{ and } y.$$

$$x + 2 = \frac{1}{2}y^3 \qquad \text{Add 2 to each side.}$$

$$2x + 4 = y^3 \qquad \text{Multiply each side by 2.}$$

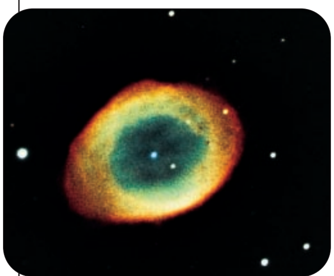
$$\sqrt[3]{2x + 4} = y \qquad \text{Take cube root of each side.}$$

▶ The inverse function is $f^{-1}(x) = \sqrt[3]{2x + 4}$.

EXAMPLE 6 *Writing an Inverse Model*

ASTRONOMY Near the end of a star's life the star will eject gas, forming a planetary nebula. The Ring Nebula is an example of a planetary nebula. The volume V (in cubic kilometers) of this nebula can be modeled by $V = (9.01 \times 10^{26})t^3$ where t is the age (in years) of the nebula. Write the inverse model that gives the age of the nebula as a function of its volume. Then determine the approximate age of the Ring Nebula given that its volume is about 1.5×10^{38} cubic kilometers.

FOCUS ON APPLICATIONS



ASTRONOMY The Ring Nebula is part of the constellation Lyra. The radius of the nebula is expanding at an average rate of about 5.99×10^8 kilometers per year.

APPLICATION LINK www.mcdougallittell.com

SOLUTION

$$V = (9.01 \times 10^{26})t^3 \qquad \text{Write original model.}$$

$$\frac{V}{9.01 \times 10^{26}} = t^3 \qquad \text{Isolate power.}$$

$$\sqrt[3]{\frac{V}{9.01 \times 10^{26}}} = t \qquad \text{Take cube root of each side.}$$

$$(1.04 \times 10^{-9})\sqrt[3]{V} = t \qquad \text{Simplify.}$$

To find the age of the nebula, substitute 1.5×10^{38} for V.

$$t = (1.04 \times 10^{-9})\sqrt[3]{V} \qquad \text{Write inverse model.}$$

$$= (1.04 \times 10^{-9})\sqrt[3]{1.5 \times 10^{38}} \qquad \text{Substitute for } V.$$

$$\approx 5500 \qquad \text{Use a calculator.}$$

▶ The Ring Nebula is about 5500 years old.

GUIDED PRACTICE

Vocabulary Check ✓

1. Explain how to use the horizontal line test to determine if an inverse relation is an inverse function.

Concept Check ✓

2. Describe how the graph of a relation and the graph of its inverse are related.

3. Explain the steps in finding an equation for an inverse function.

Skill Check ✓

Find the inverse relation.

4.

x	1	2	3	4	5
y	−1	−2	−3	−4	−5

5.

x	−4	−2	0	2	4
y	2	1	0	1	2

Find an equation for the inverse relation.

6. $y = 5x$ **7.** $y = 2x - 1$ **8.** $y = -\frac{2}{3}x + 6$

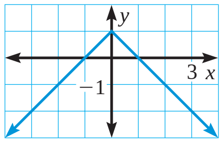
Verify that *f* and *g* are inverse functions.

9. $f(x) = 8x^3,\ g(x) = \frac{x^{1/3}}{2}$ **10.** $f(x) = 6x + 3,\ g(x) = \frac{1}{6}x - \frac{1}{2}$

Find the inverse function.

11. $f(x) = 3x^4,\ x \ge 0$ **12.** $f(x) = 2x^3 + 1$

13. The graph of $f(x) = -|x| + 1$ is shown. Is the inverse of f a function? Explain.



Ex. 13

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 949.

INVERSE RELATIONS **Find the inverse relation.**

14.

x	1	4	1	0	1
y	3	−1	6	−3	9

15.

x	1	−2	4	2	−2
y	0	3	−2	2	−1

FINDING INVERSES **Find an equation for the inverse relation.**

16. $y = -2x + 5$ **17.** $y = 3x - 3$ **18.** $y = \frac{1}{2}x + 6$

19. $y = -\frac{4}{5}x + 11$ **20.** $y = 11x - 5$ **21.** $y = -12x + 7$

22. $y = 3x - \frac{1}{4}$ **23.** $y = 8x - 13$ **24.** $y = -\frac{3}{7}x + \frac{5}{7}$

STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 14–24
Example 2: Exs. 25–32
Example 3: Exs. 57–59
Example 4: Exs. 33–41
Example 5: Exs. 42–56
Example 6: Exs. 60–62

VERIFYING INVERSES **Verify that *f* and *g* are inverse functions.**

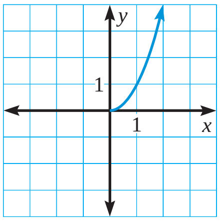
25. $f(x) = x + 7,\ g(x) = x - 7$ **26.** $f(x) = 3x - 1,\ g(x) = \frac{1}{3}x + \frac{1}{3}$

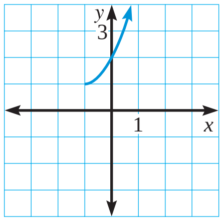
27. $f(x) = \frac{1}{2}x + 1,\ g(x) = 2x - 2$ **28.** $f(x) = -2x + 4,\ g(x) = -\frac{1}{2}x + 2$

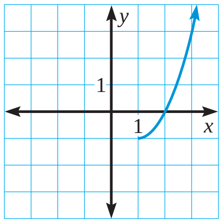
29. $f(x) = 3x^3 + 1,\ g(x) = \left(\frac{x-1}{3}\right)^{1/3}$ **30.** $f(x) = \frac{1}{3}x^2,\ x \ge 0;\ g(x) = (3x)^{1/2}$

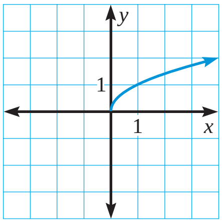
31. $f(x) = \frac{x^5 + 2}{7},\ g(x) = \sqrt[5]{7x - 2}$ **32.** $f(x) = 256x^4,\ x \ge 0;\ g(x) = \frac{\sqrt[4]{x}}{4}$

VISUAL THINKING **Match the graph with the graph of its inverse.**

33.


34.


35.


A.


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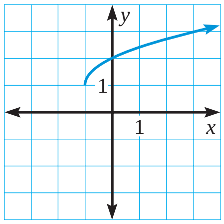
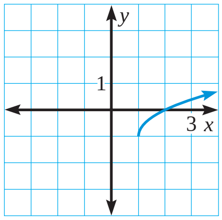


C.


INVERSES OF POWER FUNCTIONS **Find the inverse power function.**

36. $f(x) = x^7$ **37.** $f(x) = -x^6, x \geq 0$ **38.** $f(x) = 3x^4, x \leq 0$

39. $f(x) = \frac{1}{32}x^5$ **40.** $f(x) = 10x^3$ **41.** $f(x) = -\frac{9}{4}x^2, x \leq 0$

INVERSES OF NONLINEAR FUNCTIONS **Find the inverse function.**

42. $f(x) = x^3 + 2$ **43.** $f(x) = -2x^5 + \frac{1}{3}$ **44.** $f(x) = 2 - 2x^2, x \leq 0$

45. $f(x) = \frac{3}{5}x^3 - 9$ **46.** $f(x) = x^4 - \frac{1}{2}, x \geq 0$ **47.** $f(x) = \frac{1}{6}x^5 + \frac{2}{3}$

HORIZONTAL LINE TEST **Graph the function f. Then use the graph to determine whether the inverse of f is a function.**

48. $f(x) = -2x + 3$ **49.** $f(x) = x + 3$ **50.** $f(x) = x^2 + 1$

51. $f(x) = -3x^2$ **52.** $f(x) = x^3 + 3$ **53.** $f(x) = 2x^3$

54. $f(x) = |x| + 2$ **55.** $f(x) = (x + 1)(x - 3)$ **56.** $f(x) = 6x^4 - 9x + 1$

57. **EXCHANGE RATE** The Federal Reserve Bank of New York reports international exchange rates at 12:00 noon each day. On January 20, 1999, the exchange rate for Canada was 1.5226. Therefore, the formula that gives Canadian dollars in terms of United States dollars on that day is

$$D_C = 1.5226 D_{US}$$

where D_C represents Canadian dollars and D_{US} represents United States dollars. Find the inverse of the function to determine the value of a United States dollar in terms of Canadian dollars on January 20, 1999.

INTERNET **DATA UPDATE** of Federal Reserve Bank of New York data at www.mcdougallittell.com

58. **TEMPERATURE CONVERSION** The formula to convert temperatures from degrees Fahrenheit to degrees Celsius is:

$$C = \frac{5}{9}(F - 32)$$

Write the inverse of the function, which converts temperatures from degrees Celsius to degrees Fahrenheit. Then find the Fahrenheit temperatures that are equal to 29°C, 10°C, and 0°C.

FOCUS ON CAREERS

REAL LIFE **INVESTMENT BANKER**

Investment bankers have a wide variety of job descriptions. Some buy and sell international currencies at reported exchange rates, discussed in Ex. 57.

INTERNET **CAREER LINK**
www.mcdougallittell.com

59. **BOWLING** In bowling a *handicap* is a change in score to adjust for differences in players' abilities. You belong to a bowling league in which each bowler's handicap h is determined by his or her average a using this formula:

$$h = 0.9(200 - a)$$

(If the bowler's average is over 200, the handicap is 0.) Find the inverse of the function. Then find your average if your handicap is 27.

60. **GAMES** You and a friend are playing a number-guessing game. You ask your friend to think of a positive number, square the number, multiply the result by 2, and then add 3. If your friend's final answer is 53, what was the original number chosen? Use an inverse function in your solution.

61. **FISH** The weight w (in kilograms) of a hake, a type of fish, is related to its length l (in centimeters) by this function:

$$w = \left(9.37 \times 10^{-6}\right)l^3$$

Find the inverse of the function. Then determine the approximate length of a hake that weighs 0.679 kilogram. ▶ Source: *Fishbyte*

Hake

STUDENT HELP

INTERNET HOMEWORK HELP Visit our Web site www.mcdougallittell.com for help with problem solving in Ex. 62.

62. **SHELVES** The weight w (in pounds) that can be supported by a shelf made from half-inch Douglas fir plywood can be modeled by

$$w = \left(\frac{82.9}{d}\right)^3$$

where d is the distance (in inches) between the supports for the shelf. Find the inverse of the function. Then find the distance between the supports of a shelf that can hold a set of encyclopedias weighing 66 pounds.

Test Preparation

QUANTITATIVE COMPARISON **In Exercises 63 and 64, choose the statement that is true about the given quantities.**

Ⓐ The quantity in column A is greater.

Ⓑ The quantity in column B is greater.

Ⓒ The two quantities are equal.

Ⓓ The relationship cannot be determined from the given information.

	Column A	Column B
63.	$f^{-1}(3)$ where $f(x) = 6x + 1$	$f^{-1}(-4)$ where $f(x) = -2x + 9$
64.	$f^{-1}(2)$ where $f(x) = -5x^3$	$f^{-1}(0)$ where $f(x) = x^3 + 14$

★ Challenge

INVERSE FUNCTIONS **Complete Exercises 65–68 to explore functions that are their own inverses.**

65. **VISUAL THINKING** The functions $f(x) = x$ and $g(x) = -x$ are their own inverses. Graph each function and explain why this is true.

66. Graph other linear functions that are their own inverses.

67. Write equations of the lines you graphed in Exercise 66.

68. Use your equations from Exercise 67 to find a general formula for a family of linear equations that are their own inverses.

EXTRA CHALLENGE
www.mcdougallittell.com

Mixed Review

ABSOLUTE VALUE FUNCTIONS **Graph the absolute value function.** (Review 2.8 for 7.5)

69. $f(x) = |x| - 1$

70. $f(x) = 2|x| + 7$

71. $f(x) = |x - 4| + 5$

72. $f(x) = -3|x + 2| - 7$

QUADRATIC FUNCTIONS **Graph the quadratic function.** (Review 5.1 for 7.5)

73. $f(x) = x^2 + 2$

74. $f(x) = (x + 3)^2 - 7$

75. $f(x) = 2(x + 2)^2 - 5$

76. $f(x) = -3(x - 4)^2 + 1$

SIMPLIFYING EXPRESSIONS **Simplify the expression. Assume all variables are positive.** (Review 7.2)

77. $\sqrt[4]{20} \cdot \sqrt[4]{\frac{4}{5}}$

78. $\left(\frac{1}{9}\right)^{1/6}\left(\frac{1}{9}\right)^{1/3}$

79. $\frac{(5y)^{1/5}}{(5y)^{6/5}}$

80. $\sqrt[6]{2x^6}$

81. $3\sqrt[7]{5} + 2\sqrt[7]{5}$

82. $\sqrt[3]{270} + 2\sqrt[3]{10}$

83. **SNACK FOODS** Delia, Ruth, and Amy go to the store to buy snacks. Delia buys 3 bagels and 3 apples. Ruth buys 1 pretzel, 2 bagels, and 3 apples. Amy buys 2 pretzels and 4 bagels. Delia's bill comes to \$3.72, Ruth's to \$5.06, and Amy's to \$6.58. How much does one bagel cost? (Review 3.6)

Quiz 2

Self-Test for Lessons 7.3 and 7.4

Let $f(x) = 6x^2 - x^{1/2}$ and $g(x) = 2x^{1/2}$. Perform the indicated operation and state the domain. (Lesson 7.3)

1. $f(x) + g(x)$

2. $f(x) - g(x)$

3. $f(x) \cdot g(x)$

4. $\frac{f(x)}{g(x)}$

Let $f(x) = 3x^{-1}$ and $g(x) = x - 8$. Perform the indicated operation and state the domain. (Lesson 7.3)

5. $f(g(x))$

6. $g(f(x))$

7. $f(f(x))$

8. $g(g(x))$

Verify that f and g are inverse functions. (Lesson 7.4)

9. $f(x) = 2x - 3,\ g(x) = \frac{1}{2}x + \frac{3}{2}$

10. $f(x) = (x + 1)^{1/3},\ g(x) = x^3 - 1$

Find the inverse function. (Lesson 7.4)

11. $f(x) = x + 8$

12. $f(x) = 2x^4,\ x \le 0$

13. $f(x) = -x^5 + 6$

Graph the function f. Then use the graph to determine whether the inverse of f is a function. (Lesson 7.4)

14. $f(x) = 3x^6 + 2$

15. $f(x) = -2x^5 + 3x - 1$

16. $f(x) = 6\sqrt[3]{x + 4}$

17. **RIPPLES IN A POND** You drop a pebble into a calm pond causing ripples of concentric circles. The radius r (in feet) of the outer ripple is given by $r(t) = 0.6t$ where t is the time (in seconds) after the pebble hits the water. The area A (in square feet) of the outer ripple is given by $A(r) = \pi r^2$. Use composition of functions to find the relationship between area and time. Then find the area of the outer ripple after 2 seconds. (Lesson 7.3)