Chapter 4

Exponential and logarithmic functions

CHALLENGE 4

4.1 Basic exponential function

4.2 Exponential function

4.3 Basic logarithmic function

4.4 Logarithmic function

4.5 Logarithmic calculations

EVALUATION 4

CHALLENGE 4

1. We are looking for the exponent x such that $2^x = 5$.

a) Find an approximation of x, to the nearest tenth, by estimating. $x = 2.3$

b) 1. What is the exact value of x? $x = \log_2 5$

2. Use a calculator to approximate to the nearest thousandth the exact value of x.

$x = 2.322$

2. A herd presently consists of 7 elephants. This herd doubles every 6 years. After how many years will this herd contain 100 elephants?

After 23 years

$y = 7\left(2^{t/6}\right)$ $\quad \frac{100}{7} = 2$ $\quad \frac{t}{6}\log 2 = \log\frac{100}{7}$ $\quad \frac{6\log\frac{100}{7}}{\log 2} = t$ $\quad t = 23$

3. A \$1000 capital is invested for 5 years at an annual interest rate of 10%, compounded twice per year. Determine the accumulated capital.

$1000(1 + 0.05)^{10} = \$1628.89$

4. A ball bounces $\frac{3}{5}$ the height of its previous bounce. The ball is dropped from a 25 m high building. What height does the ball reach on the sixth bounce?

1.17 m

5. A full grocery cart for 4 people presently costs \$125. If the rate of inflation remains constant at 3.6% over the next 10 years, in how many years will this cart cost \$160?

In 7 years $\quad 160 = 125(1.036)^x \quad x = 6.98$

6. Raphaelle pays \$24 000 for a new car. Two years later, its value is evaluated at \$13 500. After how many years will the car be worth \$5695?

After 5 years

7. On January 1st 2000, a village had a population of 1500 residents. On January 1st 2007, this same village had a population of 1050 residents. If the rate of decay remains constant, in what year will the population of this village reach 800 residents?

$1500 \cdot c^7 = 1050$; $c \approx 0.95$; $1500(0.95)^t = 800$; $t = 12.26$. In 2012.

8. The value of a house, bought for \$150 000, has increased at a constant rate of 8% per year since its purchase. How long after its purchase will this house be worth more than \$250 000?

$150\,000(1.08)^t > 250\,000$; $t > 6.64$. It will take at least 6.64 years.

6. $y = ac^x$

$13500 = ac^2$

$24000 = ac^0$

$a = 24000$

$\frac{13500}{24000} = c^2$

$c = 0.75$

$5695 = 24000(0.75)^x$

$x = \frac{\log .237}{\log .75}$

$x = 5$

4.1 Basic exponential function

Activity 1 Exponential growth

The number of cells in a controlled environment doubles every day. Initially ($x = 0$), we observed one million cells.

x	y
–2	*0.25*
–1	*0.5*
0	1
1	*2*
2	*4*
3	*8*

a) Complete the table of values for the function that gives the number y of cells (in millions) as a function of the number x of elapsed days since the beginning of the experiment.

b) What is the rule of this function? *$y = 2^x$*

c) How many cells are observed

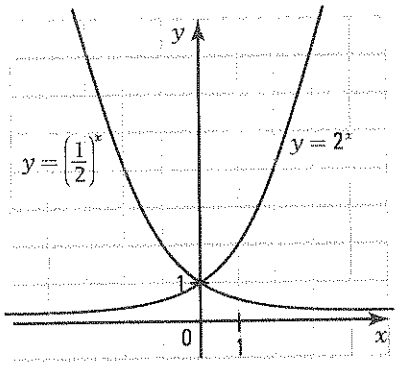
1. 5 days after the beginning? *32 million*
2. 3 days prior to the beginning? *125 000*

d) The number of cells grows rapidly. The growth is said to be **exponential**. At what moment do we observe

1. 128 million cells? *7 days after the beginning*
2. 62 500 cells? *4 days prior to the beginning*

Activity 2 Basic exponential function

Consider the functions $f(x) = 2^x$ and $g(x) = \left(\frac{1}{2}\right)^x$ where $x \in \mathbb{R}$.



a) Complete the corresponding table of values for each function.

1.

x	–2	–1	0	1	2	3
$y = 2^x$	$\frac{1}{4}$	$\frac{1}{2}$	*1*	*2*	*4*	*8*

2.

x	–2	–1	0	1	2	3
$y = \left(\frac{1}{2}\right)^x$	*4*	*2*	*1*	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

b) Graph each function in the Cartesian plane.

c) Complete the table below which gives the properties of these two functions.

	$f(x) = 2^x$	$f(x) = \left(\frac{1}{2}\right)^x$
Dom f	$\mathbb{R}$	$\mathbb{R}$
Ran f	$\mathbb{R}_+^*$	$\mathbb{R}_+^*$
Zero	*none*	*none*
Initial value	*1*	*1*
$f \geq 0$ over	$\mathbb{R}$	$\mathbb{R}$
$f < 0$ over	*never*	*never*
$f \nearrow$ over	$\mathbb{R}$	*never*
$f \searrow$ over	*never*	$\mathbb{R}$
Extrema	*none*	*none*

We observe that each of the curves representing these functions gets closer and closer to the x axis without ever touching it.

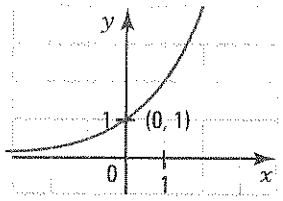
*We say that the x-axis is an **asymptote** to the curve or that the curve is asymptotic to the x-axis.*

EXPONENTIAL FUNCTION $f(x) = c^x$

- The function $f(x) = c^x$, where c is a positive real number different from 1, is called the basic **exponential** function in **base** c.

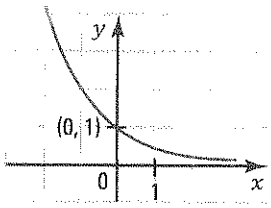
  This function describes situations of **exponential growth** or **decay**, depending on the base being greater or less than 1.

$c > 1$



Increasing function

$0 < c < 1$



Decreasing function

- Regardless of the base, we have:
  - dom $f = \mathbb{R}$ and ran $f = \mathbb{R}^*_+$.
  - The initial value is equal to 1 ($c^0 = 1$).
  - The function has no zero.
  - The function is positive over $\mathbb{R}$.
  - The x-axis is an **asymptote** to the curve.

- When the independent variable x increases by 1 unit, the dependent variable y is multiplied by the **multiplicative factor** c (basic exponential function) called **periodic multiplicative factor**.

$+1 \quad +1 \quad +1$

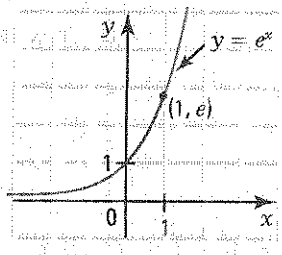
x	0	1	2	3	...
y	1	c	c^2	c^3	...

$\times c \quad \times c \quad \times c$

- The most common bases of the exponential function are the numbers 10 and e.

  e is an **irrational** number occurring in phenomena related to physics, biology, probability, ...

$$e = 2.71828...$$



1. Given the exponential function $f(x) = c^x$. Explain why

a) the base c is different from 1. ***$f(x) = 1^x = 1$ The curve $f(x)$ is a horizontal line that cannot describe a situation involving exponential growth or decay.***

b) the base c is different from 0. ***$f(x) = 0^x = 0$ ($x \neq 0$), same reason as a)***

c) the base c cannot be negative. ***c^x is not always defined when $c < 0$. Ex.: $(-4)^{\frac{1}{2}} = \sqrt{-4} \notin \mathbb{R}$***

2. For each of the following exponential functions,

1. determine the base.
2. indicate if the function is increasing or decreasing.

a) $f(x) = \left(\frac{4}{3}\right)^x$ $\frac{4}{3}$; *increasing*

b) $f(x) = \left(\frac{4}{5}\right)^x$ $\frac{4}{5}$; *decreasing*

c) $f(x) = 2^{-x}$ $\frac{1}{2}$; *decreasing*

d) $f(x) = \left(\frac{2}{3}\right)^{-x}$ $\frac{3}{2}$; *increasing*

e) $f(x) = 3^{2x}$ 9; *increasing*

f) $f(x) = \left(\frac{9}{4}\right)^{-\frac{x}{2}}$ $\frac{2}{3}$; *decreasing*

3. In each of the following cases, the point A belongs to the graph of an exponential function defined by the rule $y = c^x$. Determine the rule of each function.

a) A(2, 9) $y = 3^x$

b) $A\left(\frac{1}{2}, 2\right)$ $y = 4^x$

c) $A\left(-\frac{1}{2}, \frac{3}{2}\right)$ $y = \left(\frac{4}{9}\right)^x$

d) $A\left(-\frac{1}{3}, \frac{1}{2}\right)$ $y = 8^x$

e) $A\left(-\frac{1}{2}, \frac{5}{4}\right)$ $y = \left(\frac{16}{25}\right)^x$

f) $A\left(-\frac{3}{2}, \frac{8}{27}\right)$ $y = \left(\frac{9}{4}\right)^x$

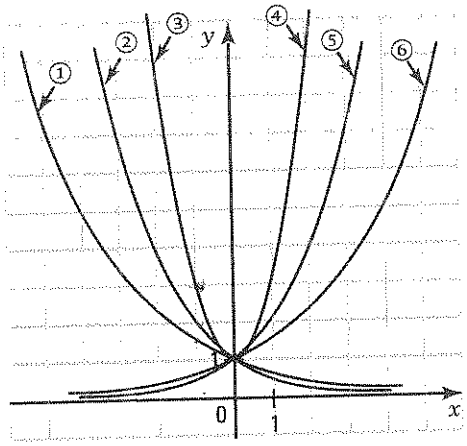
4. The point $A\left(-\frac{1}{2}, \frac{2}{3}\right)$ belongs to the graph of an exponential function defined by the rule $y = c^x$.

a) A point B on this graph has an x-coordinate of -2. What is its y-coordinate? $\left(\frac{9}{4}\right)^{-2} = \frac{16}{81}$

b) A point C on this graph has a y-coordinate of $\frac{4}{9}$. What is its x-coordinate? -1

5. On the right, we have represented the exponential functions defined by the rules:

$y = 2^x, y = \left(\frac{3}{2}\right)^x, y = 3^x, y = \left(\frac{1}{2}\right)^x, y = \left(\frac{2}{3}\right)^x, y = \left(\frac{1}{3}\right)^x.$



a) Associate each curve to its equation.

1. $y = \left(\frac{2}{3}\right)^x$ 2. $y = \left(\frac{1}{2}\right)^x$ 3. $y = \left(\frac{1}{3}\right)^x$

4. $y = 3^x$ 5. $y = 2^x$ 6. $y = \left(\frac{3}{2}\right)^x$

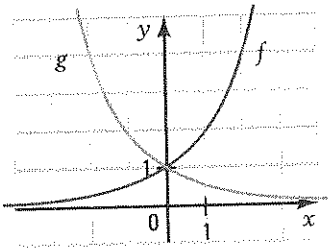
b) Of the three increasing exponential functions, which one increases the fastest? Justify your answer.

$y = 3^x$. It is the one with the biggest base.

c) Of the three decreasing exponential functions, which one decreases the fastest? Justify your answer.

$y = \left(\frac{1}{3}\right)^x$. It is the one with the smallest base.

6. The functions $f(x) = 2^x$ and $g(x) = \left(\frac{1}{2}\right)^x$ are represented on the right.



a) Verify that $f(2) = g(-2)$. *$f(2) = 4$; $g(-2) = 4$.*

b) 1. Show that $f(x) = g(-x)$ for any real number x.

$g(-x) = \left(\frac{1}{2}\right)^{-x} = 2^x = f(x)$. $\forall x \in \mathbb{R}$

2. What can be deduced from the curves of the exponential functions representing the functions f and g?

The curves are symmetrical about the y-axis.

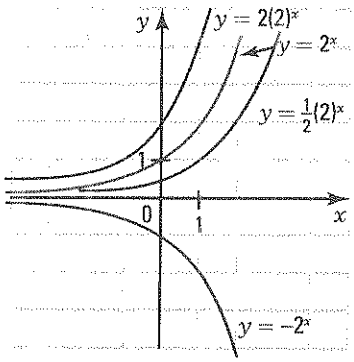
4.2 Exponential function

ACTIVITY 1 Exponential function $f(x) = ac^x$ – Role of parameter a

a) Consider the basic exponential function $f(x) = 2^x$ and the transformed exponential function $g(x) = a(2)^x$.

Represent, in the same Cartesian plane, the functions $g_1(x) = 2(2)^x$, $g_2(x) = \frac{1}{2}(2)^x$ and $g_3(x) = -2^x$ and explain how to deduce the graph of $g(x)$ from the graph of $f(x)$ when

1. $a > 1$: ***By a vertical stretch.***
2. $0 < a < 1$: ***By a vertical reduction.***
3. $a = -1$: ***By a reflection about the x-axis.***



b) From the graph of $y = 2^x$, we get the graph of $y = a(2)^x$ by the transformation $(x, y) \rightarrow$ ***(x, ay)***.

c) The accumulated value $c(t)$ after t years of an invested capital a at an interest rate i compounded annually is calculated using the rule:

$$\boxed{c(t) = a\,(1 + i)^t}$$

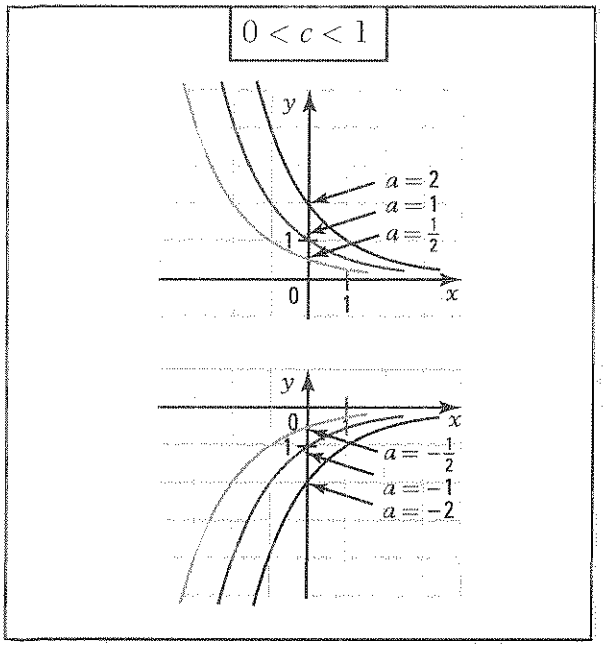
A capital of \$1000 is invested in a bank at an interest rate of 8% compounded annually.

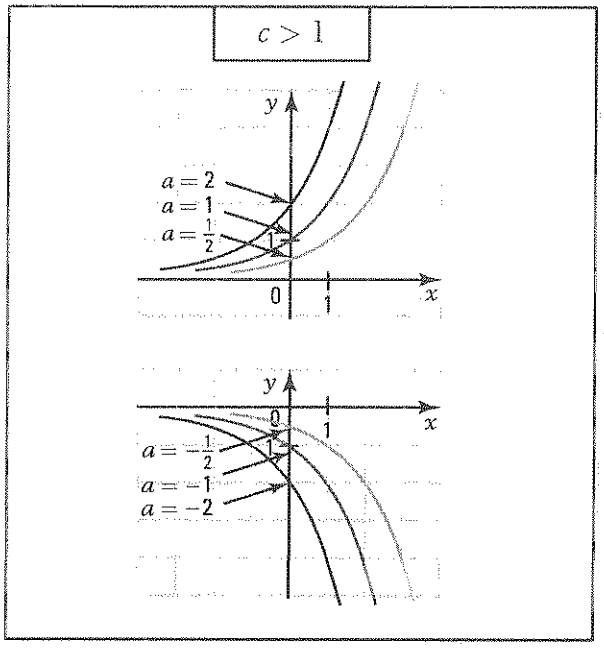
x	y
0	1000
1	1080
2	***1166.40***
3	***1259.71***
⋮	

1. Complete the table of values which associates the number t of years to the accumulated capital $c(t)$.
2. What is the rule of this exponential function? ***$c(t) = 1000(1.08)^t$***
3. What is the base of the exponential function? ***1.08***
4. Determine and interpret the role of the parameter a.
   $a = 1000$. a represents the initial capital ($t = 0$), $c(0) = a = \$1000$.

EXPONENTIAL FUNCTION $f(x) = ac^x$ – ROLE OF PARAMETER a

- The exponential curve undergoes a **vertical stretch** when the absolute value of parameter a increases.
- The parameter a corresponds to the **initial value** of the function.





- The horizontal scale change $(x, y) \rightarrow (x, ay)$ transforms the graph of $y = c^x$ into a graph defined by the rule $y = ac^x$.

1. Consider the function $f(x) = ac^x$.

a) What is the domain of f? $\mathbb{R}$

b) What is the range of f when

1. $a > 0$? $\mathbb{R}_+^*$ 2. $a < 0$? $\mathbb{R}_-^*$

c) Does the function f have a zero? ***No***

d) Does the function f have an asymptote? If yes, what is the equation of the asymptote?

Yes, the x-axis with equation y = 0.

e) Indicate if the function is increasing or decreasing when

1. $a > 0$ and $c > 1$. ***increasing*** 2. $a > 0$ and $0 < c < 1$. ***decreasing***

3. $a < 0$ and $c > 1$. ***decreasing*** 4. $a < 0$ and $0 < c < 1$. ***increasing***

2. For each of the following functions, indicate if

1. the function is positive or negative.
2. the function is increasing or decreasing.

a) $y = -3\left(\frac{1}{4}\right)^x$

1. ***Negative***
2. ***Increasing***

b) $y = 2\left(\frac{3}{2}\right)^x$

1. ***Positive***
2. ***Increasing***

c) $y = -\frac{1}{4}(2)^x$

1. ***Negative***
2. ***Decreasing***

d) $y = 10\left(\frac{1}{2}\right)^x$

1. ***Positive***
2. ***Decreasing***

e) $y = 2(3)^{-x}$
1. *Positive*
2. *Decreasing*

f) $y = -2(5)^{-x}$
1. *Negative*
2. *Increasing*

g) $y = \left(\frac{1}{3}\right)^{-x}$
1. *Positive*
2. *Increasing*

h) $y = -\left(\frac{3}{2}\right)^{-x}$
1. *Negative*
2. *Increasing*

3. Julie invests an amount of \$1000 in a bank at an interest rate of 8% compounded annually. The capital, $C(t)$, accumulated after t months, is given by $C(t) = 1000(1.08)^t$. What is the accumulated capital after

a) 3 years? *\$1259.71* b) 5 years? *\$1469.33*

4. A radioactive substance disintegrates over time. Its mass m (in grams) is expressed as a function of time t (in years) by the equation $m = 10(0.8)^t$. What is the mass of this substance

a) today ($t = 0$)? *10 g* b) in 2 years? *6.4 g* c) one year ago? *12.5 g*

5. The value $v(t)$ of a car that depreciates by 20% per year is given by $v(t) = v_0(0.80)^t$ where v_0 represents the initial value of the car and t represents the number of years since its purchase.

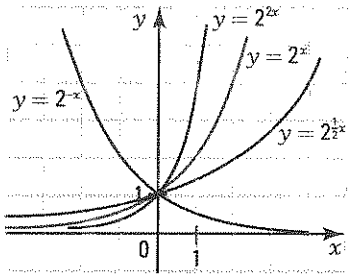
a) What is the value of a new car 3 years after its purchase if it was bought for \$30 000?
\$15 360

b) How much did you pay for a car now worth \$22 400 two years after its purchase? *\$35 000*

ACTIVITY 2 Exponential function $f(x) = c^{bx}$ — Role of parameter b

a) Consider the basic exponential function $f(x) = 2^x$ and the transformed exponential function $g(x) = 2^{bx}$.

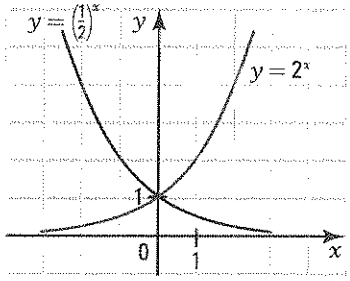
Represent, in the same Cartesian plane, the functions $g_1(x) = 2^{2x}$, $g_2(x) = 2^{\frac{1}{2}x}$ and $g_3(x) = 2^{-x}$ and explain how to deduce the graph of $g(x)$ from the graph of $f(x)$ when



1. $b > 1$: *By a horizontal reduction.*
2. $0 < b < 1$: *By a horizontal stretch.*
3. $b = -1$: *By a reflection about the y-axis.*

b) From the graph of $y = 2^x$, we get the graph of $y = 2^{bx}$ by the transformation $(x, y) \rightarrow$ $\left(\frac{x}{b}, y\right)$.

c) The graph of the exponential function $y = 2^x$ is represented on the right.



1. Draw the graph of the function $y = \left(\frac{1}{2}\right)^x$.

2. Compare the graph of the function $y = \left(\frac{1}{2}\right)^x$ and that of the function $y = 2^{-x}$, found in a).

Justify algebraically. *They are identical,* $\left(\frac{1}{2}\right)^x = (2^{-1})^x = 2^{-x}$.

d) The value $c(t)$ of a capital a accumulated after t years invested at an interest rate i compounded n times per year is calculated according to the rule:

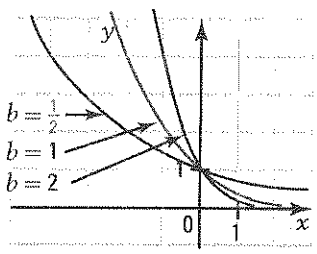
$$\boxed{c(t) = a\left(1 + \frac{i}{n}\right)^{nt}}$$

1. We invest \$1 at an interest rate of 12% compounded monthly. Establish the rule of the exponential function which gives the accumulated capital $c(t)$. ***$c(t) = (1.01)^{12t}$***
2. What is the base of the exponential function? ***1.01***
3. Determine and interpret the role of parameter b. ***$b = 12$. The parameter b indicates the number of times that interest is deposited per year.***

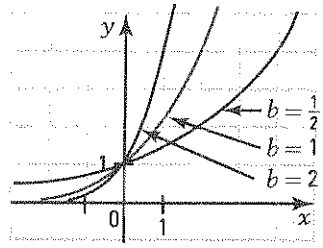
EXPONENTIAL FUNCTION $f(x) = c^{bx}$ – ROLE OF PARAMETER b

- When we increase parameter b, the exponential curve undergoes a **horizontal reduction** regardless of its base.
- If x represents elapsed time, the parameter b corresponds to the number of periods per unit of time.

$0 < c < 1$



$c > 1$



- The vertical scale change $(x, y) \rightarrow \left(\frac{x}{b}, y\right)$ transforms the graph of $y = c^x$ into a graph defined by the rule $y = c^{bx}$.

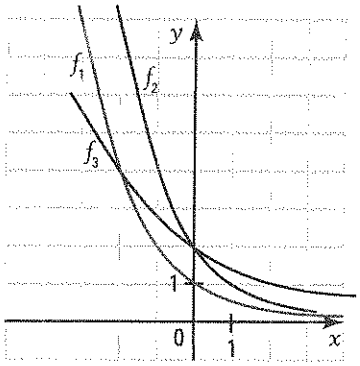
6. Consider the following exponential functions:

$f_1(x) = \left(\frac{1}{2}\right)^x, f_2(x) = 2\left(\frac{1}{2}\right)^x, f_3(x) = 2\left(\frac{1}{2}\right)^{\frac{x}{2}}$

a) What geometric transformation applies

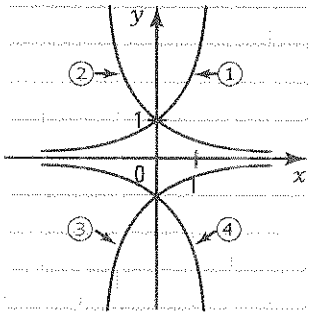
1. the graph of f_1 onto the graph of f_2?
   The vertical stretch $(x, y) \rightarrow (x, 2y)$
2. the graph of f_2 onto the graph of f_3?
   The horizontal stretch $(x, y) \rightarrow (2x, y)$

b) Using the geometric transformations established in a), draw the graphs of the functions f_2 and f_3.



7. a) Among the four functions represented on the right, identify

1. $y = e^x$ ① 2. $y = -e^x$ ③
3. $y = e^{-x}$ ② 4. $y = -e^{-x}$ ④



b) Complete the table below.

	$y = e^x$	$y = -e^x$	$y = e^{-x}$	$y = -e^{-x}$
Domain	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
Range	$\mathbb{R}_+^*$	$\mathbb{R}_-^*$	$\mathbb{R}_+^*$	$\mathbb{R}_-^*$
Sign	*Positive*	*Negative*	*Positive*	*Negative*
Variation	*Increasing*	*Decreasing*	*Decreasing*	*Increasing*

8. The number of bacteria, in a controlled environment, doubles on average every 15 minutes. Initially, this environment contains one thousand cells. Consider the function which gives the number N(t) of cells as a function of the number t of hours elapsed since the beginning.

t	N(t)
0	*1*
0.25	*2*
0.50	*4*
0.75	*8*
1	*16*
2	*256*

a) Complete the table of values on the right.

b) What is the rule of the function? $y = 2^{4t}$

c) What is the number of cells after 3 hours? *4096 thousand*

d) After how many hours does this environment contain 1024 thousand cells? *2 h 30 min*

9. Mr. Desmond invests \$30 000 in a bank that offers a 4.2% annual interest rate. Using the formula given in activity 4, determine the accumulated amount Mr. Desmond receives after 5 years if the interest is compounded

$y = 30000\left(1 + \frac{0.042}{x}\right)^{5x}$

a) annually. *\$36 851.90* $n = 1$

b) every 3 months. *\$36 969.84* $n = 4$ $\lim_{x \to \infty} y = 30000(e)^{.042(5)}$

c) every 4 months. *\$36 956.48* $n = 3$ $= 37010.3418$

d) every 6 months. *\$36 929.95* $n = 2$

e) monthly. *\$36 996.77* $n = 12$

10. At the birth of his son Raphael, Alex invests \$2000 in an education savings bond. This account's annual interest of 3.7% is compounded monthly. What will be the accumulated value of this bond when Raphael turns 20 years old?

\$4187.10

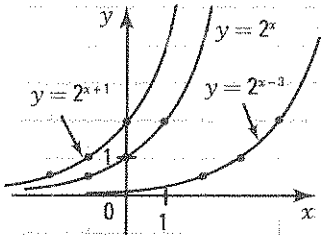
$\hookrightarrow = 2000\left(1 + \frac{0.037}{12}\right)^{12 \cdot 20}$

$= \$4187.10$

ACTIVITY 3 Exponential function $f(x) = c^{x-h} + k$ — Role of parameters h and k

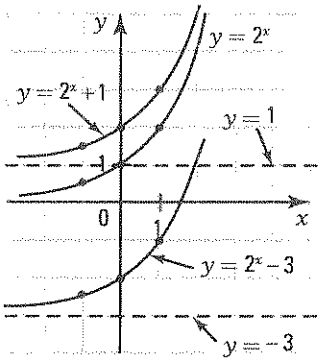
a) The basic exponential function $y = 2^x$ is represented on the right..

1. Explain how to deduce the graph of
   1) $y = 2^{x-3}$ ***A horizontal translation of 3 units to the right.***
   2) $y = 2^{x+1}$ ***A horizontal translation of 1 unit to the left.***
2. By applying the role of parameter h to the critical points $\left(-1, \frac{1}{2}\right)$, $(0, 1)$ and $(1, 2)$ of the basic function, graph the functions
   1) $y = 2^{x-3}$ 2) $y = 2^{x+1}$



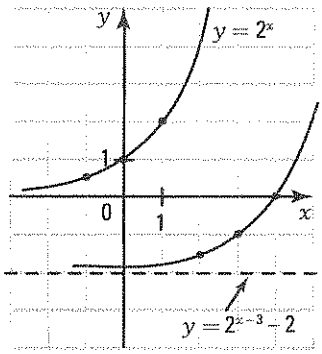
b) The basic exponential function $y = 2^x$ is represented on the right..

1. Explain how to deduce the graph of
   1) $y = 2^x - 3$ ***A vertical translation of 3 units downward.***
   2) $y = 2^x + 1$ ***A vertical translation of 1 unit upward.***
2. By applying the role of parameter k to the critical points $\left(-1, \frac{1}{2}\right)$, $(0, 1)$ and $(1, 2)$ of the basic function, graph the functions
   1) $y = 2^x - 3$ 2) $y = 2^x + 1$
3. Determine the equation of the asymptote of the function
   1) $y = 2^x - 3$ ***y = −3*** 2) $y = 2^x + 1$ ***y = 1***



c) The basic exponential function $y = 2^x$ and the transformed exponential function $y = 2^{x-3} - 2$ are represented on the right.

Complete: The graph of the function $y = 2^{x-3} - 2$ is obtained from the graph of the function $y = 2^x$ by ***a horizontal translation of 3 units to the right followed by a vertical translation of 2 units downward.***



ACTIVITY 4 Exponential function $f(x) = ac^{b(x-h)} + k$

The basic exponential function $y = c^x$ can be transformed into an exponential function of the form $f(x) = ac^{b(x-h)} + k$.

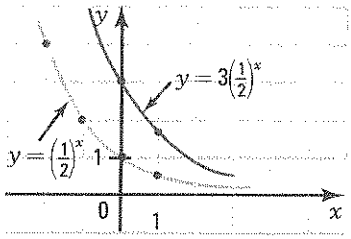
Consider the exponential function $y = \left(\frac{1}{2}\right)^x$ and the exponential function $y = 3\left(\frac{1}{2}\right)^{2(x-1)} - 3$, each with the same base $\frac{1}{2}$.

a) Identify the parameters a, b, h and k. ***a = 3, b = 2, h = 1, k = –3***

b) The introduction of the parameters a, b, h and k in the rule of the basic exponential function causes a series of transformations to the graph. Complete the description of this process.

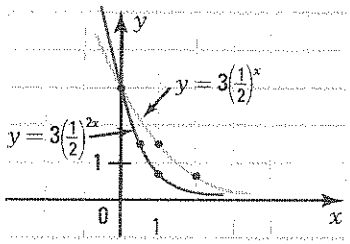
1. From the graph $y = \left(\frac{1}{2}\right)^x$, we obtain the graph of $y = 3\left(\frac{1}{2}\right)^x$ by the transformation $(x, y) \rightarrow$ ***(x, 3y)***
   The parameter a causes ***a vertical stretch of factor 3.***

1.



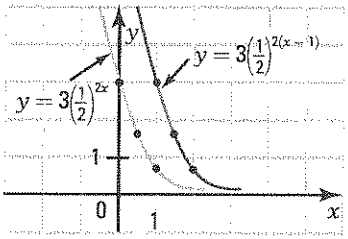
2. From the graph $y = 3\left(\frac{1}{2}\right)^x$, we obtain the graph of $y = 3\left(\frac{1}{2}\right)^{2x}$ by the transformation $(x, y) \rightarrow$ $\left(\frac{x}{2}, y\right)$
   The parameter b causes ***a horizontal reduction of factor $\frac{1}{2}$.***

2.



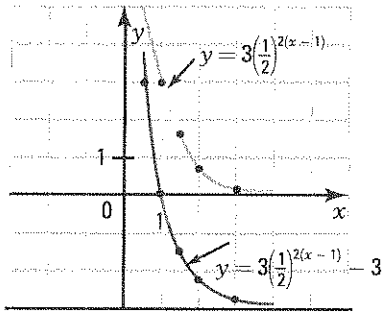
3. From the graph $y = 3\left(\frac{1}{2}\right)^{2x}$, we obtain the graph of $y = 3\left(\frac{1}{2}\right)^{2(x-1)}$ by the transformation $(x, y) \rightarrow$ ***(x + 1, y)***
   The parameter h causes ***a horizontal translation of 1 unit to the right.***

3.



4. From the graph $y = 3\left(\frac{1}{2}\right)^{2(x-1)}$, we obtain the graph of $y = 3\left(\frac{1}{2}\right)^{2(x-1)} - 3$ by the transformation $(x, y) \rightarrow$ ***(x, y – 3)***
   The parameter k causes ***a vertical translation of 3 units downward.***

4.



c) What does the line $y = -3$ represent for the function $y = 3\left(\frac{1}{2}\right)^{2(x-1)} - 3$?

The asymptote of the function.

d) Verify, using the tables of values, that the transformation $(x, y) \rightarrow \left(\frac{x}{b} + h, ay + k\right)$ directly applies the graph of $y = \left(\frac{1}{2}\right)^x$ onto the graph of $y = 3\left(\frac{1}{2}\right)^{2(x-1)} - 3$.

$y = \left(\frac{1}{2}\right)^x$

x	y
-1	2
0	1
1	$\frac{1}{2}$

$y = 3\left(\frac{1}{2}\right)^{2(x-1)} - 3$

x	y
$\frac{1}{2}$	3
1	0
$\frac{3}{2}$	$-\frac{3}{2}$

Indeed, (–1, 2) becomes $\left(\frac{1}{2}, 3\right)$

(0, 1) becomes (1, 0) and $\left(1, \frac{1}{2}\right)$ ***becomes*** $\left(\frac{3}{2}, -\frac{3}{2}\right)$

by the transformation $(x, y) \rightarrow \left(\frac{x}{2} + 1, 3y - 3\right)$.

EXPONENTIAL FUNCTION $f(x) = ac^{b(x-h)} + k$

- The graph of the function $f(x) = ac^{b(x-h)} + k$ is deduced from the graph of the basic exponential function $y = c^x$ by the transformation

$$\boxed{(x, y) \rightarrow \left(\frac{x}{b} + h, ay + k\right)}$$

- The exponential function $f(x) = ac^{b(x-h)} + k$ has the horizontal line $y = k$ as an asymptote.
  Ex.: See activity 4.

11. The following functions have a rule of the form $f(x) = a(2)^{b(x-h)} + k$.

$f_1(x) = 3(2)^x$; $f_2(x) = 2^{3x}$; $f_3(x) = 2^{x+3}$; $f_4(x) = 2^x + 3$; $f_5(x) = -3\,(2)^{\frac{1}{2}(x+4)} - 5$;

Complete the table on the right by determining, for each function, the parameters a, b, h and k and by giving the rule of the transformation which enables you to obtain the graph of the function from the graph of the basic function $f(x) = 2^x$.

	a	b	h	k	Rule
f_1	3	1	0	0	$(x, y) \rightarrow (x, 3y)$
f_2	1	3	0	0	$(x, y) \rightarrow \left(\frac{x}{3}, y\right)$
f_3	1	1	–3	0	$(x, y) \rightarrow (x - 3, y)$
f_4	1	1	0	3	$(x, y) \rightarrow (x, y + 3)$
f_5	–3	$\frac{1}{2}$	–4	–5	$(x, y) \rightarrow (2x - 4, -3y - 5)$

12. In each of the following cases, a transformation is applied to the basic exponential function $y = 5^x$.

Find the rule of the function whose graph is obtained by the following transformations.

a) $(x, y) \rightarrow (x, -y)$ $y = -5^x$

b) $(x, y) \rightarrow (x - 2, y - 3)$ $y = 5^{x+2} - 3$

c) $(x, y) \rightarrow \left(\frac{x}{2}, y\right)$ $y = 5^{2x}$

d) $(x, y) \rightarrow (2x, y)$ $y = 5^{\frac{x}{2}}$

e) $(x, y) \rightarrow (3x, -2y)$ $y = -2(5)^{\frac{x}{3}}$

f) $(x, y) \rightarrow \left(\frac{x}{3} + 1, 2y - 1\right)$ $y = 2(5)^{3(x-1)} - 1$

13. Write the rules of the following exponential functions in the form $y = ac^{b(x-h)} + k$.

a) $y = 2^{3x-6}$ $y = 2^{3(x-2)}$

b) $y = 5^{-2x+6} + 1$ $y = 5^{-2(x-3)} + 1$

c) $y = 5(3)^{2x+1}$ $y = 5(3)^{2\left(x+\frac{1}{2}\right)}$

d) $y = 3\left(\frac{1}{2}\right)^{4-2x} - 5$ $y = 3\left(\frac{1}{2}\right)^{-2(x-2)} - 5$

ACTIVITY 5 Exponential equation – Form $c^u = c^v$

a) Justify the steps in solving the equation $3(2)^{4x} = 768$.

$3(2)^{4x} = 768$

$\Leftrightarrow \quad 2^{4x} = 256$ ***Divide each side by 3.***

$\Leftrightarrow \quad 2^{4x} = 2^8$ ***Write the right side as a power of 2.***

$\Leftrightarrow \quad 4x = 8$ ***Deduce the equality of the exponents.***

$\Leftrightarrow \quad x = 2$ ***Deduce the value of the unknown x.***

b) The number $P(t)$ of bacteria in a controlled environment is given by $P(t) = 100(2)^{3t}$ where t represents the number of elapsed days since the beginning.

How many days after the beginning do we observe 6400 cells?

$100(2)^{3t} = 6400$

$2^{3t} = 64$

$2^{3t} = 2^6$

$t = 2$. ***We observe 6400 cells 2 days after the beginning of the experiment.***

EXPONENTIAL EQUATION – FORM $c^u = c^v$

When both sides of an exponential equation can be written as powers of the same base, we use the logical equivalence:

$$c^u = c^v \Leftrightarrow u = v$$

	Ex.: $5(2)^{3x} = 320$
1. Isolate the power containing the unknown on one side.	$2^{3x} = 64$
2. Write the equation as powers of the same base.	$2^{3x} = 2^6$
3. Make the exponents equal.	$3x = 6$
4. Determine the value of the unknown.	$x = 2$
5. Establish the solution set S.	Thus, $S = \{2\}$.

14. Solve the following exponential equations.

a) $3^x = 243$
$x = 5$

b) $2^x = \frac{1}{8}$
$x = -3$

c) $2(5)^x = 250$
$x = 3$

d) $5^{2x} - 1 = 0$
$x = 0$

e) $2(5)^x - 48 = 2$
$x = 2$

f) $9^x - 27 = 0$
$x = \frac{3}{2}$

g) $3(4)^x - 96 = 0$
$x = \frac{5}{2}$

h) $\frac{1}{2}(8)^x - 16 = 0$
$x = \cancel{\frac{8}{3}}$ $5/3$

i) $27\left(\frac{4}{9}\right)^x - 8 = 0$
$x = \frac{3}{2}$

15. Determine the zero, if it exists, of the following exponential functions.

a) $y = 5(3)^{x-2} - 15$

3

b) $y = 2(3)^{-(x+2)} - 18$

-4

c) $y = -3\left(\frac{1}{2}\right)^{-2(x+3)} + 12$

-2

d) $y = -5\left(\frac{1}{5}\right)^{x-1} + 4$

e) $y = -4\left(\frac{2}{3}\right)^{x-1} + 9$

-1

f) $y = 3\left(\frac{2}{5}\right)^{-2(x+1)} + \frac{12}{25}$

No zero

16. Solve the following exponential equations.

a) $2^{3x} \cdot 2^{2x} = \frac{1}{4}$

$x = -\frac{2}{5}$

b) $\frac{3^x}{3^{2x}} = 27$

$x = -3$

c) $\left(\frac{1}{2}\right)^x = 16$

$x = -4$

d) $2^x \cdot 2^x = 64$

$x = 3$

e) $(2^x)^2 = 16(2)^x$

$x = 4$

f) $2^{x^2} = 16$

$x = -2$ *or* $x = 2$

17. The growth of a herd of bison follows the rule $P(t) = P_0 \times 2^{\frac{t}{10}}$ where P_0 represents the initial population and $P(t)$ the population after t years. In how many years will the bison population quadruple its initial population?

$4\,P_0 = P_0(2)^{\frac{t}{10}} \Leftrightarrow 4 = 2^{\frac{t}{10}} \Leftrightarrow t = 20$. *After 20 years.*

18. A mosquito population doubles every seven days. If there were 5 mosquitoes initially, after how many days will the population contain 80 mosquitoes?

$5(2)^{\frac{t}{7}} = 80 \Leftrightarrow 2^{\frac{t}{7}} = 16 \Leftrightarrow t = 28$. *After 28 days.*

19. A 100 g radioactive mass disintegrates according to the rule $m(t) = 100\left(\frac{1}{2}\right)^{\frac{t}{4}}$ where $m(t)$ is the resulting mass after t hours.

a) Determine after how many hours the resulting mass is equal to 25 g. *8 hours*

b) We call the **half-life** of a radioactive substance the time necessary for its mass to be reduced by half by disintegration. What is the half-life of this mass? *4 hours*

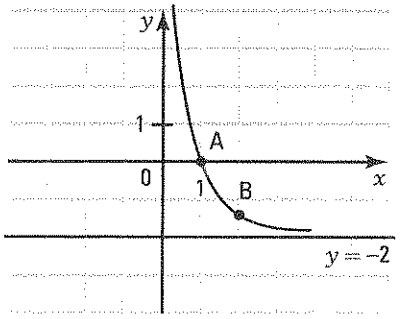
ACTIVITY 6 Sketch of the graph and study of an exponential function

To sketch the graph of the function $f(x) = ac^{b(x-h)} + k$,

1. we draw the horizontal asymptote defined by the equation $y = k$.
2. we locate, using the rule, 2 points on the exponential curve.
3. we sketch the exponential curve.

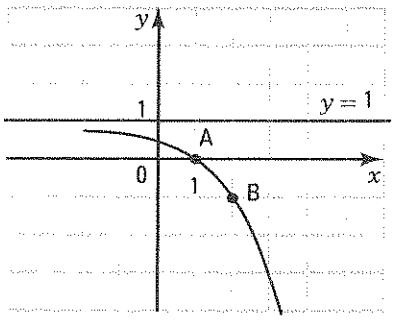
a) Sketch the graphs of the following functions.

1. $f_1(x) = 2\left(\frac{1}{2}\right)^{2(x-1)} - 2$



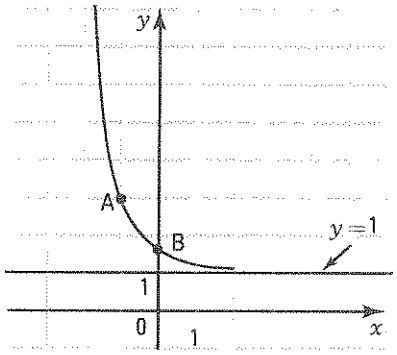
A(1, 0); B(2, −1.5)

2. $f_2(x) = -\frac{1}{4}(2)^{x+1} + 1$



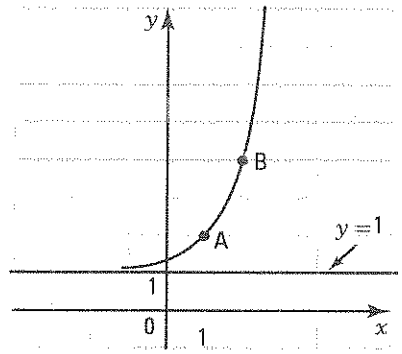
A(1, 0); B(2, −1)

3. $f_3(x) = 2\left(\frac{1}{2}\right)^{2(x+1)} + 1$



A(−1, 3); B(0, 1.5)

4. $f_4(x) = 3^{x-1} + 1$



A(1, 2); B(2, 4)

b) Find the zero, when it exists, of the preceding functions.

1. f_1: *1* 2. f_2: *1* 3. f_3: *No zero* 4. f_4: *No zero*

c) Do a study of each of the preceding functions and complete the following table.

	a	b	h	k	Asymptote	Domain	Range	Zero	Sign	Variation
f_1	2	2	1	−2	$y = -2$	$\mathbb{R}$	$]-2, +\infty[$	1	$f(x) \geq 0$ if $x \in]-\infty, 1]$ $f(x) \leq 0$ if $x \in [1, +\infty[$	Decreasing over $\mathbb{R}$
f_2	$-\frac{1}{4}$	1	−1	1	$y = 1$	$\mathbb{R}$	$]-\infty, 1[$	1	$f(x) \geq 0$ if $x \in]-\infty, 1]$ $f(x) \leq 0$ if $x \in [1, +\infty[$	Decreasing over $\mathbb{R}$
f_3	2	2	−1	1	$y = 1$	$\mathbb{R}$	$]1, +\infty[$	None	$f(x) > 0, \forall x \in \mathbb{R}$	Decreasing over $\mathbb{R}$
f_4	1	1	1	1	$y = 1$	$\mathbb{R}$	$]1, +\infty[$	None	$f(x) > 0, \forall x \in \mathbb{R}$	Increasing over $\mathbb{R}$

d) Verify that, for any exponential function, $f(x) = ac^{b(x-h)} + k$,

1. $\text{dom} f = \mathbb{R}$
2. the asymptote is the line $y = k$.
3. $\text{ran} f =]k, +\infty[$ if $a > 0$ and $\text{ran} f =]-\infty, k[$ if $a < 0$.
4. the zero of f exists if a and k are opposite signs.

STUDY OF AN EXPONENTIAL FUNCTION

- To sketch the graph of an exponential function,
  1. we draw the horizontal asymptote.
  2. we locate, using the rule, two points on the exponential curve.
  3. we sketch the exponential curve.
- We study an exponential function from the sketch of the graph. We have:
  - $\text{dom} f = \mathbb{R}$.
  - $\text{ran} f =]k, +\infty[$ if $a > 0$ and $\text{ran} f =]-\infty, k[$ if $a < 0$.
  - The zero of f exists if a and k are opposite signs.
  - The sign of f is determined from the graph.
  - The variation of f is determined from the graph.
  - The function f has no extrema.

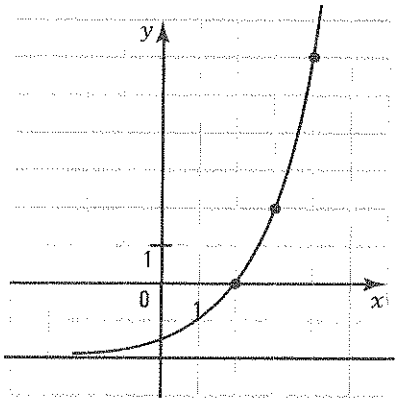
  Ex.: See activity 6.

20. For each of the following exponential functions, determine

1. the equation of the asymptote.
2. a sketch of the graph.
3. the domain.
4. the range.
5. the zero.
6. the sign.
7. the variation.

Rule	1	2	3	4	5	6	7
$f_1(x) = 2^{x-1} - 4$	$y = -4$	$(1, -3)$; $(2, -2)$	$\mathbb{R}$	$]-4, +\infty[$	3	$f_1(x) \leq 0$ if $x \in]-\infty, 3]$ $f_1(x) \geq 0$ if $x \in [3, +\infty[$	Increasing
$f_2(x) = 4\left(\frac{1}{2}\right)^{x+1} + 1$	$y = 1$	$(-1, 5)$; $(1, 2)$	$\mathbb{R}$	$]1, +\infty[$	None	$f_2(x) > 0 \ \forall\, x \in \mathbb{R}$	Decreasing
$f_3(x) = -2^{-(x-1)} - 1$	$y = -1$	$(0, -3)$; $(1, -2)$	$\mathbb{R}$	$]-\infty, -1[$	None	$f_3(x) < 0 \ \forall\, x \in \mathbb{R}$	Increasing
$f_4(x) = 2^{-(x-1)} - 4$	$y = -4$	$(0, -2)$; $(1, -3)$	$\mathbb{R}$	$]-4, +\infty[$	-1	$f_4(x) \geq 0$ if $x \in]-\infty, -1]$ $f_4(x) \leq 0$ if $x \in [-1, +\infty[$	Decreasing

21. Consider the function $f(x) = 2(4)^{\frac{1}{2}(x-2)} - 2$.



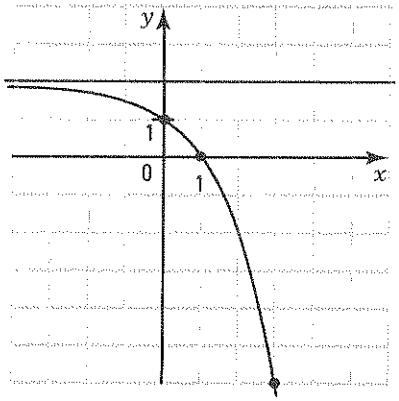
a) Represent the function f in the Cartesian plane using the table of values.

x	-2	-1	0	1	2	3	4
y	$-\frac{15}{8}$	$-\frac{7}{4}$	$-\frac{3}{2}$	-1	0	2	6

b) Determine

1. dom f $\mathbb{R}$
2. ran f $]-2, +\infty[$
3. the zero of f 2
4. the sign of f. $f(x) \leq 0$ if $x \leq 2$; $f(x) \geq 0$ if $x \geq 2$
5. the variation of f. f is increasing over $\mathbb{R}$.
6. the equation of the asymptote. $y = -2$

22. Consider the function $f(x) = -2\left(\frac{1}{4}\right)^{-\frac{1}{2}(x-1)} + 2$.



a) Represent the function f in the Cartesian plane using the table of values.

x	-3	-2	-1	0	1	2	3
y	$\frac{15}{8}$	$\frac{7}{4}$	$\frac{3}{2}$	1	0	-2	-6

b) Determine

1. dom f $\mathbb{R}$
2. ran f $]-\infty, 2[$
3. the zero of f 1
4. the sign of f. $f(x) \geq 0$ if $x \leq 1$; $f(x) \leq 0$ if $x \geq 1$
5. the variation of f. f is decreasing over $\mathbb{R}$.
6. the equation of the asymptote. $y = 2$

23. Study the sign of the following exponential functions.

a) $f(x) = \left(\frac{1}{2}\right)^{x+2} - 4$

$f(x) \leq 0$ if $x \in [-4, +\infty[$

$f(x) \geq 0$ if $x \in]-\infty, -4]$

b) $f(x) = -3(2)^{x-1} + 12$

$f(x) \leq 0$ if $x \in [3, +\infty[$

$f(x) \geq 0$ if $x \in]-\infty, 3]$

c) $f(x) = -2(3)^{-2(x+1)} + 18$

$f(x) \leq 0$ if $x \in]-\infty, -2]$

$f(x) \geq 0$ if $x \in [-2, +\infty[$

24. Solve the following exponential inequalities.

a) $-2^{x-2} + 8 \geq 0$ — $S =]-\infty, 5]$

b) $6(3)^{x-2} - 2 > 0$ — $S =]1, +\infty[$

c) $3(2)^{\frac{1}{2}(x+1)} - 12 \leq 0$ — $S =]-\infty, 3]$

25. What is the range of the function $f(x) = -5(3)^{2(x-1)} + 10$? $ran f =]-\infty, 10[$

26. Determine the initial value of $f(x) = -3(4)^{\frac{1}{2}(x+1)} + 10$. 4

27. Study the variation of the following functions.

a) $f(x) = -2^{x-1} + 4$ — *Decreasing*

b) $f(x) = -2\left(\frac{1}{3}\right)^{x+1} + 6$ — *Increasing*

ACTIVITY 7 Rule of an exponential function: $f(x) = ac^{bx}$

a) In a reforestation project, a company decides to double the number of trees it plants each year. Initially, the forest contained 25 trees.

1. Find the rule of the function which gives the number y of trees in the forest as a function of the number x of elapsed years. $y = 25(2)^x$
2. How many trees will there be after 7 years? 3200 trees

b) A controlled environment contains 3 bacteria initially. The number of bacteria doubles every 15 minutes. We are trying to find the rule of the exponential function which gives the number y of bacteria as a function of the elapsed time in hours. The form of the rule is: $y = ac^{bx}$.

1. Complete the table of values on the right.

x	y
0	3
$\frac{1}{4}$	6
$\frac{1}{2}$	12
$\frac{3}{4}$	24
1	48
2	758

2. The initial value of the function represented by a is the value of y at the beginning ($x = 0$). What is the initial value a? $a = 3$
3. The number of bacteria doubles every 15 minutes. What is the multiplicative factor c? $c = 2$
4. Since the elapsed time x is in hours, the parameter b represents the number of 15-minute periods per hour.
   1) What is the value of parameter b? $b = 4$
   2) What is the rule of the function in this situation? $y = 3(2)^{4x}$
5. What would the rule be if the bacteria doubles
   1) every twenty minutes? $y = 3(2)^{3x}$
   2) every two hours? $y = 3(2)^{\frac{x}{2}}$

RULE OF AN EXPONENTIAL FUNCTION $f(x) = ac^{bx}$

Once the unit of time is well defined, the independent variable x represents the elapsed time since the beginning ($x = 0$). The dependent variable y (number of bacteria, accumulated capital, ...) takes values that are periodically multiplied by a constant factor.

a represents the initial value of the dependent variable.
b represents the number of periods per unit of time.
c represents the periodic multiplicative factor, which is the factor that the dependent variable y is multiplied by each period.

Ex.: A controlled environment initially contains 10 bacteria. The unit of time is in hours.

- When the number of bacteria triples ($c = 3$) every 15 minutes, each hour contains 4 periods ($b = 4$). The rule is $y = 10(3)^{4x}$.
- When the number of bacteria doubles ($c = 2$) every 30 minutes, each hour contains 2 periods ($b = 2$). The rule is $y = 10(2)^{2x}$.
- When the number of bacteria quadruples ($c = 4$) every 2 hours, each hour contains $\frac{1}{2}$ of a period $\left(b = \frac{1}{2}\right)$. The rule is $y = 10(4)^{\frac{1}{2}x}$.

28. The tables of values below correspond to an exponential function with a rule of the form $y = ac^x$. Find the rule of each function.

a)

x	0	1
y	$\frac{1}{2}$	$\frac{3}{2}$

$y = \frac{1}{2}(3)^x$

b)

x	0	−1
y	2	8

$y = 2\left(\frac{1}{4}\right)^x$

c)

x	0	2
y	−3	−12

$y = -3(2)^x$

d)

x	0	−2
y	−4	−9

$y = -4\left(\frac{2}{3}\right)^x$

29. Each of the following situations is described by an exponential function of the form $y = ac^{bx}$. After establishing the unit of time,

1. define the variables x and y. 2. determine the parameters a, b and c. 3. find the rule of the function.

a) In a controlled environment containing 1000 bacteria initially, the number of bacteria triples every 10 minutes.

1. *x: number of elapsed hours since the beginning, y: number of bacteria*
2. $a = 1000$, $b = 6$, $c = 3$ 3. $y = 1000(3)^{6x}$

b) In an environment initially containing 100 insects, the number of insects doubles every 3 days.

1. *x: number of elapsed days since the beginning, y: number of insects*
2. $a = 100$, $b = \frac{1}{3}$, $c = 2$ 3. $y = 100(2)^{\frac{1}{3}x}$

c) A car bought for $30 000 loses 20% of its value every year.

1. *x: number of elapsed years since the purchase, y: value of the car*
2. $a = 30\ 000$, $b = 1$, $c = 0.80$ 3. $y = 30\ 000(0.80)^x$

d) An initial population of 1000 deer increases by 15% each year.

1. *x: number of elapsed years since the beginning, y: deer population*
2. $a = 1000$, $b = 1$, $c = 1.15$ 3. $y = 1000(1.15)^x$

e) A radioactive mass of 50 g loses half of its mass each period of 6 hours starting at noon.

1. *x: number of elapsed hours since noon, y: remaining mass*

2. $a = 50$; $b = \frac{1}{6}$, $c = \frac{1}{2}$ 3. $y = 50\left(\frac{1}{2}\right)^{\frac{x}{6}}$

f) An initial population of 100 birds increases by 15% every 2 years.

1. *x: number of elapsed ~~hours~~ years since the beginning, y: bird population*

2. $a = 100$, $b = \frac{1}{2}$, $c = 1.15$ 3. $y = 100(1.15)^{\frac{x}{2}}$

30. A capital c_0 is invested at a fixed annual interest rate i compounded n times per year. The accumulated capital $c(t)$ after t years is given by the formula

$$c(t) = c_0\left(1 + \frac{i}{n}\right)^{nt}.$$

a) Establish the rule of the function $c(t)$ that gives the accumulated capital of \$1000 invested at a 6% interest rate compounded

1. annually. $C(t) = 1000\,(1.06)^t$ 2. every 6 months. $C(t) = 1000(1.03)^{2t}$

3. each month. $C(t) = 1000(1.005)^{12t}$ 4. each day. $C(t) = 1000\left(1 + \frac{0.06}{365}\right)^{365t}$

b) Calculate the accumulated capital after 5 years in each of the preceding cases.

1. \$1338.23 2. \$1343.92 3. \$1348.85 4. \$1349.83

ACTIVITY 8 Finding the rule of an exponential function

a) Consider the exponential functions $f(x) = a(c)^{bx}$ and $g(x) = a(c^b)^x$.

1. Identify

1) the base of f. c 2) the base of g. c^b

2. Explain why the functions f and g have the same rule.

$(c^b)^x = c^{bx}$ *Law of exponents:* $(a^m)^n = a^{mn}$

b) Consider the exponential functions $f(x) = ac^{x-h}$ and $g(x) = Ac^x$.
Determine A for which the functions f and g have the same rule.

$ac^{x-h} = ac^x \cdot c^{-h} = ac^{-h}c^x = Ac^x$; $A = ac^{-h}$.

c) Write each of the following rules in the form $y = ac^x + k$.

1. $y = 5(2)^{3x} + 1$ $y = 5(2)^{3x} + 1 = 5(2^3)^x + 1 = 5(8)^x + 1$

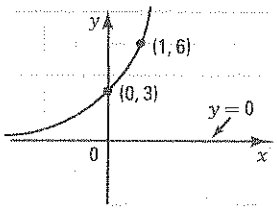
2. $y = 3(2)^{x+2} + 5$ $y = 3 \cdot 2^x \cdot 2^2 + 5 = 12(2)^x + 5$

3. $y = 2(3)^{2x+1} - 4$ $y = 2 \cdot 3^{2x} \cdot 3 - 4 = 6(9)^x - 4$

Therefore, every exponential function can be represented by a rule of the form $y = ac^x + k$.

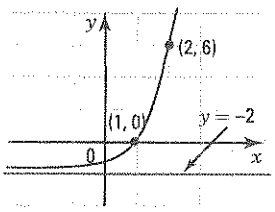
d) Find the rule of the following exponential functions knowing the asymptote and two points on the graph.

1.



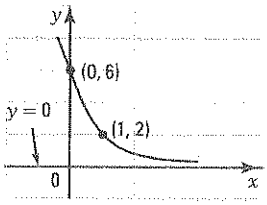
$y = 3(2)^x$

2.



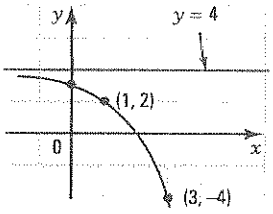
$y = \frac{1}{2}(4)^x - 2$

3.



$y = 6\left(\frac{1}{3}\right)^x$

4.



$y = -2^x + 4$

FINDING THE RULE OF AN EXPONENTIAL FUNCTION

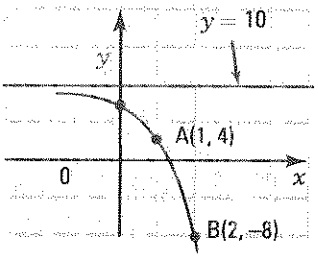
Every rule of an exponential function can be written in the form $y = ac^x + k$.

Ex.: Consider the exponential function f on the right.

- Horizontal asymptote: $y = 10 \Rightarrow k = 10$.
- $f(1) = 4$
- $f(2) = -8$

$$\Rightarrow \begin{cases} ac + 10 = 4 \\ ac^2 + 10 = -8 \end{cases} \Rightarrow \begin{cases} ac = -6 \\ ac^2 = -18 \end{cases}$$

We deduce that $c = 3$ and $a = -2$. Therefore, $f(x) = -2(3)^x + 10$.



31. Find the rule of the following exponential functions knowing the asymptote and two points, A and B, on the graph.

a) Asymptote: $y = 0$
A(1, 6); B(2, 18)
$y = 2(3)^x$

b) Asymptote: $y = 0$
A(1, −10); B(2, −50)
$y = -2(5)^x$

c) Asymptote: $y = 2$
A(0, 1); B(1, −2)
$y = -4^x + 2$

d) Asymptote: $y = -10$
A(1, −4); B(2, 8)
$y = 2(3)^x - 10$

32. A ping pong table is 70 cm high. A ball is dropped from above the table. The height reached by the ball relative to the floor is 100 cm on the first bounce and 88 cm on the second bounce.

a) Find the rule of the exponential function that gives the height y of the ball as a function of the number x of bounces.

x	1	2
y(cm)	100	88

$y = ac^x + 70$; $y = 50(0.60)^x + 70$

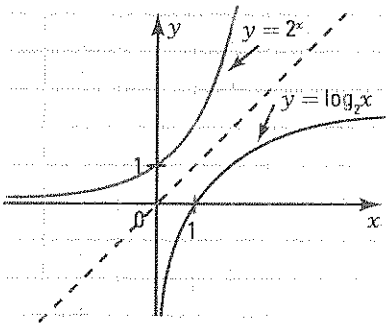
b) Calculate, to the nearest tenth of cm, the height reached by the ball on the 6th bounce.

72.3 cm

4.3 Basic logarithmic function

ACTIVITY 1 Basic logarithmic function

a) The basic exponential function $y = 2^x$ is represented on the right.



1. Graph its inverse.
2. Explain why the inverse is a function.

   Any vertical line intersects the graph at most once.

The inverse of an exponential function in base 2 is a function called ***logarithmic function in base 2****, written* $y = \log_2 x$.

3. The table of values for the exponential function $y = 2^x$ is represented on the right.

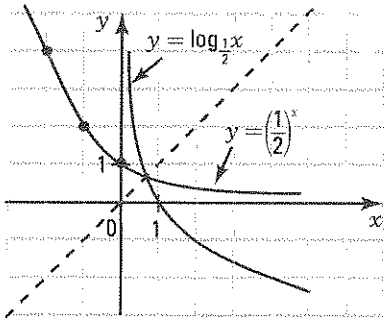
   Deduce the table of values for the logarithmic function $y = \log_2 x$.

x	-2	-1	0	1	2
$y = 2^x$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$y = \log_2 x$	-2	-1	0	1	2

4. Determine the following, and justify your answer.
   1) $\log_2 8 =$ ***3 since $2^3 = 8$***
   2) $\log_2 \frac{1}{8} =$ ***-3 since $2^{-3} = \frac{1}{8}$***
5. For the function $y = \log_2 x$, determine
   1) the domain. $\mathbb{R}_+^*$ 2) the range. $\mathbb{R}$ 3) the zero. ***1***
   4) the sign ***$\log_2 x \geq 0$ if $x \geq 1$ and $\log_2 x \leq 0$ if $0 < x \leq 1$.***
   5) the extrema, if it exists. ***No extrema*** 6) the variation. ***f is increasing over $\mathbb{R}_+^*$***
   7) the equation of the asymptote. ***$x = 0$***

b) The basic exponential function $y = \left(\frac{1}{2}\right)^x$ is represented on the right.



1. Graph its inverse.
2. Explain why the inverse is a function.

   Any vertical line intersects the graph at most once.

The inverse of an exponential function in base $\frac{1}{2}$ *is a function called* ***logarithmic function in base*** $\frac{1}{2}$*, written* $y = \log_{\frac{1}{2}} x$.

3. The table of values for the exponential function $y = \left(\frac{1}{2}\right)^x$ is represented on the right.

   Deduce the table of values for the logarithmic function $y = \log_{\frac{1}{2}} x$.

x	-2	-1	0	1	2
$y = \left(\frac{1}{2}\right)^x$	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$

x	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$
$y = \log_{\frac{1}{2}} x$	-2	-1	0	1	2

4. Determine the following, and justify your answer.

1) $\log_{\frac{1}{2}} \frac{1}{8} =$ ***3 since*** $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ 2) $\log_{\frac{1}{2}} 8 =$ ***−3 since*** $\left(\frac{1}{2}\right)^{-3} = 8$

5. For the function $y = \log_{\frac{1}{2}} x$, determine

1) the domain. $\mathbb{R}_+^*$ 2) the range. $\mathbb{R}$ 3) the zero. ***1***

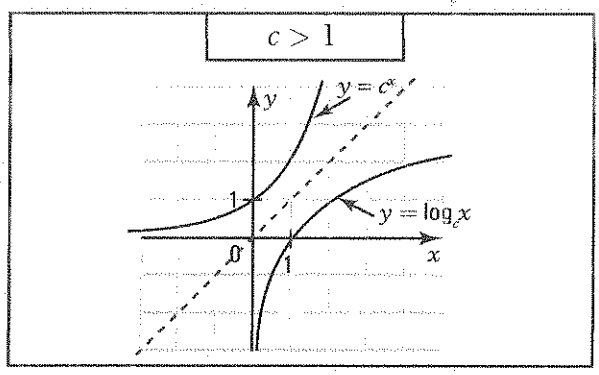
4) the sign $\log_{\frac{1}{2}} x \geqslant 0$ ***if*** $0 < x \leqslant 1$ ***and*** $\log_{\frac{1}{2}} x \leqslant 0$ ***if*** $x \geqslant 1$.

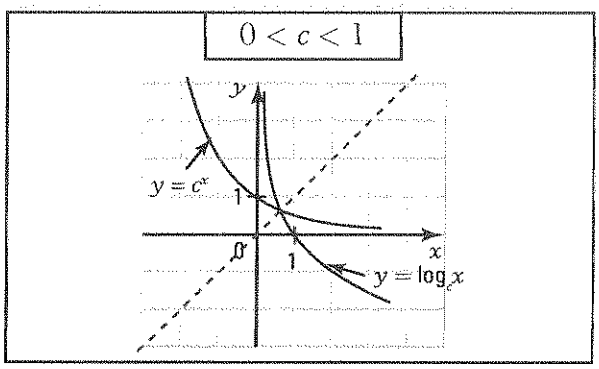
5) the extrema, if it exists. ***No extrema*** 6) the variation. ***f is decreasing over*** $\mathbb{R}_+^*$

7) the equation of the asymptote. $x = 0$

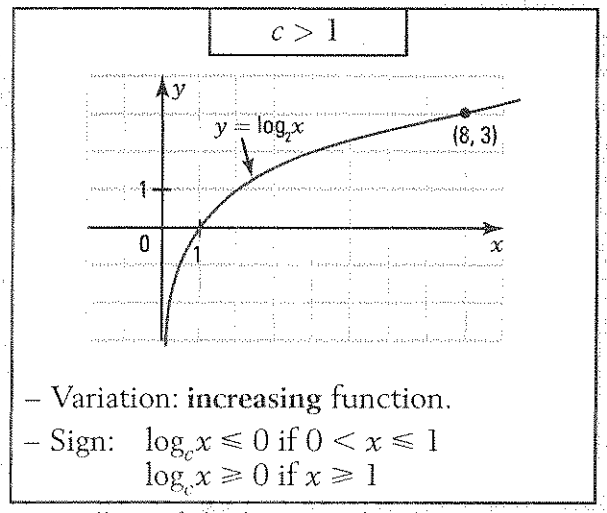
BASIC LOGARITHMIC FUNCTION $y = \log_c x$

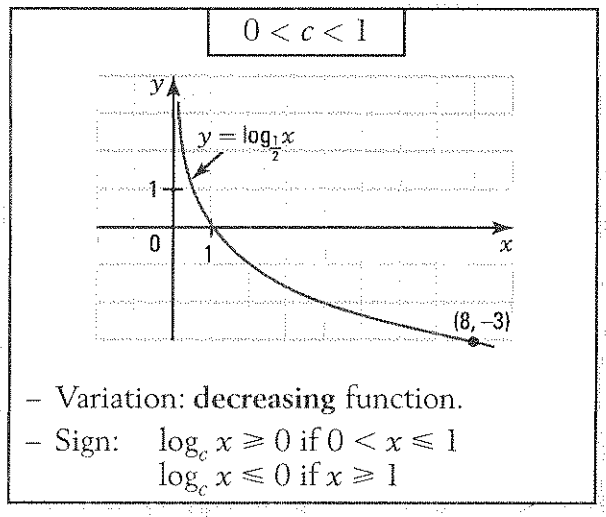
- The inverse of the basic exponential function in base c, $y = c^x$, is a function called **logarithmic function** in base c written $y = \log_c x$.





- We have: $\boxed{y = \log_c x \Leftrightarrow x = c^y}$.
- The graphical representation of $y = \log_c x$ depends on the base c.





Regardless of the base, we have:
- dom $= \mathbb{R}_+^*$ and ran $= \mathbb{R}$.
- the zero of the function is equal to 1.
- the function has no extrema.
- the function has a vertical asymptote: $x = 0$.

ACTIVITY 2 Definition of a logarithm

The logarithm in base c of a positive number p, written $\log_c p$, is the exponent q such that $c^q = p$. Thus, $\log_5 25 = 2$ since $5^2 = 25$.

We have the equivalence: $\boxed{\log_c p = q \Leftrightarrow c^q = p}$

Exponential form	Logarithmic form
$2^3 = 8$	$\log_2 8 = 3$
$3^2 = 9$	$\log_3 9 = 2$
$5^2 = 25$	$\log_5 25 = 2$
$2^{-2} = \frac{1}{4}$	$\log_2 \frac{1}{4} = -2$
$\left(\frac{2}{3}\right)^{-3} = \frac{27}{8}$	$\log_{\frac{2}{3}} \frac{27}{8} = -3$

a) Use this equivalence to complete the table on the right.

b) Calculate and justify your answer.

1. $\log_2 16 =$ *4 since $2^4 = 16$*
2. $\log_2 \frac{1}{8} =$ *-3 since $2^{-3} = \frac{1}{8}$*
3. $\log_2 1 =$ *0 since $2^0 = 1$*
4. $\log_{\frac{2}{3}} \frac{4}{9} =$ *2 since $\left(\frac{2}{3}\right)^2 = \frac{4}{9}$*

c) The **logarithm in base 10** of a number x, called a common logarithm, is written **log$x$**. Mentally evaluate the following logarithms and then verify the value on your calculator.

1. log 10. *1*
2. log 1000. *3*
3. log 0.1. *-1*
4. log 0.01. *-2*

d) The logarithm in base e of a number x, called a natural logarithm, is written **ln$x$**. Mentally evaluate the following logarithms and then verify the value on your calculator.

1. ln e. *1*
2. ln e^2. *2*
3. ln e^{-1}. *-1*

DEFINITION OF A LOGARITHM

- The **logarithm in base c** of a positive number p, written $\log_c p$, is the exponent q such that $c^q = p$.

  We have the equivalence: $\boxed{\log_c p = q \Leftrightarrow c^q = p}$

  A logarithm is an exponent.

  Ex.: $\log_2 8 = 3$ since $2^3 = 8$
- – The logarithm in base 10 of a number x is written $\log x$.
  – The logarithm in base e of a number x is written $\ln x$.
- The following formulas enable you to calculate $\log_c x$ using a calculator.

$$\boxed{\log_c x = \frac{\log x}{\log c}} \quad \text{or} \quad \boxed{\log_c x = \frac{\ln x}{\ln c}}$$

Use the log key or ln key on the calculator.

Thus, $\log_2 8 = \frac{\log 8}{\log 2} = 3$ or $\log_2 8 = \frac{\ln 8}{\ln 2} = 3$.

These formulas are established on page 178 (change of base law).

1. For each of the following exponential forms, write its equivalent logarithmic form.

a) $2^3 = 8$ *$\log_2 8 = 3$*

b) $10^2 = 100$ *$\log 100 = 2$*

c) $5^{-2} = \frac{1}{25}$ *$\log_5 \left(\frac{1}{25}\right) = -2$*

d) $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$ *$\log_{\frac{1}{2}} \left(\frac{1}{4}\right) = 2$*

e) $\left(\frac{2}{3}\right)^{-2} = \frac{9}{4}$ *$\log_{\frac{2}{3}} \left(\frac{9}{4}\right) = -2$*

2. For each of the following logarithmic forms, write its equivalent exponential form.

a) $\log_5 25 = 2$ *$5^2 = 25$*

b) $\log 1000 = 3$ *$10^3 = 1000$*

c) $\log_3 \left(\frac{1}{27}\right) = -3$ *$3^{-3} = \frac{1}{27}$*

d) $\log_{\frac{1}{5}} \left(\frac{1}{25}\right) = 2$ *$\left(\frac{1}{5}\right)^2 = \frac{1}{25}$*

e) $\log_{\frac{3}{2}} \left(\frac{8}{27}\right) = -3$ *$\left(\frac{3}{2}\right)^{-3} = \frac{8}{27}$*

3. Evaluate the following logarithms mentally, and then verify on your calculator.

a) $\log_2 16$ 4 b) $\log_5 25$ 2 c) $\log_{\frac{1}{2}} 2$ −1

d) $\log 0.001$ −3 e) $\log_{\frac{2}{3}}\left(\frac{4}{9}\right)$ 2 f) $\log_{\frac{2}{5}}\left(\frac{25}{4}\right)$ −2

g) $\log_{25} 5$ $\frac{1}{2}$ h) $\log_8 2$ $\frac{1}{3}$ i) $\log_{16}\left(\frac{1}{4}\right)$ $-\frac{1}{2}$

4. Evaluate the following logarithms mentally.

a) $\log_3 9 =$ 2 b) $\log_9 3 =$ $\frac{1}{2}$ c) $\log_{\frac{2}{3}}\left(\frac{8}{27}\right) =$ 3

d) $\log_{\frac{2}{5}}\left(\frac{25}{4}\right) =$ −2 e) $\log_8 2 =$ $\frac{1}{3}$ f) $\log_{\frac{4}{9}}\left(\frac{2}{3}\right) =$ $\frac{1}{2}$

g) $\log_{\frac{16}{9}}\left(\frac{3}{4}\right) =$ $-\frac{1}{2}$ h) $\log_{\frac{2}{3}} 1 =$ 0 i) $\log_{\frac{2}{3}}\left(\frac{3}{2}\right) =$ −1

5. Evaluate the following logarithms mentally.

a) $\log 10\,000$ 4 b) $\log 0.01$ −2 c) $\log \frac{1}{10}$ −1

d) $\ln e^{-2}$ −2 e) $\log 10^5$ 5 f) $\ln 1$ 0

6. Explain why, for any base c, we have:

a) $\log_c 1 = 0.$ since $c^0 = 1$ b) $\log_c c = 1.$ since $c^1 = c$

7. Determine x in each of the following.

a) $\log_2 x = 4$ $x = 16$ b) $\log_x 25 = 2$ $x = 5$ c) $\log_2\left(\frac{1}{8}\right) = x$ $x = -3$

d) $\log_x\left(\frac{4}{9}\right) = 2$ $x = \frac{2}{3}$ e) $\log_3 x = -2$ $x = \frac{1}{9}$ f) $\log_x \frac{25}{9} = -2$ $x = \frac{3}{5}$

g) $\log_{\frac{3}{2}} x = -3$ $x = \frac{8}{27}$ h) $\log_{\frac{2}{3}} \frac{3}{2} = x$ $x = -1$ i) $\log_5 x = 0$ $x = 1$

8. For each of the following logarithmic functions,

1. indicate the base; 2. determine the variation; 3. determine the sign.

a) $y = \log_{\frac{3}{2}} x$ 1. $\frac{3}{2}$ 2. Increasing 3. $\log_{\frac{3}{2}} x \leq 0$ if $0 < x \leq 1$; $\log_{\frac{3}{2}} x \geq 0$ if $x \geq 1$

b) $y = \log_{\frac{2}{3}} x$ 1. $\frac{2}{3}$ 2. Decreasing 3. $\log_{\frac{2}{3}} x \geq 0$ if $0 < x \leq 1$; $\log_{\frac{2}{3}} x \leq 0$ if $x \geq 1$

c) $y = \log x$ 1. 10 2. Increasing 3. $\log x \leq 0$ if $0 < x \leq 1$; $\log x \geq 0$ if $x \geq 1$

d) $y = \ln x$ 1. e 2. Increasing 3. $\ln x \leq 0$ if $0 < x \leq 1$; $\ln x \geq 0$ if $x \geq 1$

9. In each of the following cases, the point A is on the graph of a logarithmic function defined by the rule $y = \log_c x$. Determine the rule of each function.

a) A(9, 2) $y = \log_3 x$ b) A(100, 2) $y = \log x$ c) A(8, −3) $y = \log_{\frac{1}{2}} x$

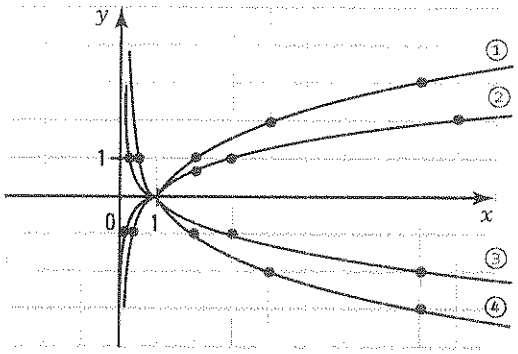
d) $A\left(\frac{27}{8}, 3\right)$ $y = \log_{\frac{3}{2}} x$ e) A(e, 1) $y = \ln x$ f) $A\left(\frac{25}{4}, -2\right)$ $y = \log_{\frac{2}{5}} x$

10. The point $A\left(2, \frac{1}{2}\right)$ is on the graph of a logarithmic function defined by the rule $y = \log_c x$.

a) A point B on this graph has an x-coordinate of 16. What is its y-coordinate? *2*

b) A point C on this graph has a y-coordinate of 3. What is its x-coordinate? *64*

11. The logarithmic functions $y = \log_2 x$, $y = \log_3 x$, $y = \log_{\frac{1}{2}} x$ and $y = \log_{\frac{1}{3}} x$ are represented on the right.



a) Find the equation associated with each curve.

1. $y = \log_2 x$ 2. $y = \log_3 x$

3. $y = \log_{\frac{1}{3}} x$ 4. $y = \log_{\frac{1}{2}} x$

b) True or False?

1. When considering two increasing logarithmic functions, the one that increases the fastest is the one with smallest base. *True*
2. When considering two decreasing logarithmic functions, the one that decreases the fastest is the one with biggest base. *True*

12. True or False?

a) $\log_c x$ is not defined when $x \leq 0$. *True*

b) $y = \log_c x$ is increasing when $c > 1$. *True*

c) For any base c, the curve defined by the rule $y = \log_c x$ passes through $(1, 0)$. *True*

d) If $0 < c < 1$ and $x > 1$ then $\log_c x > 0$. *False*

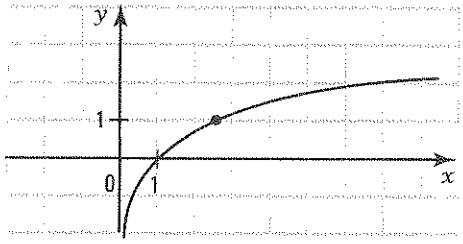
e) If $c > 1$ and $x < 1$ then $\log_c x < 0$. *True*

13. Consider the function $y = \ln x$.

a) Complete the table of values.

x	e^{-1}	1	e	e^2
$y = \ln x$	*-1*	*0*	*1*	*2*

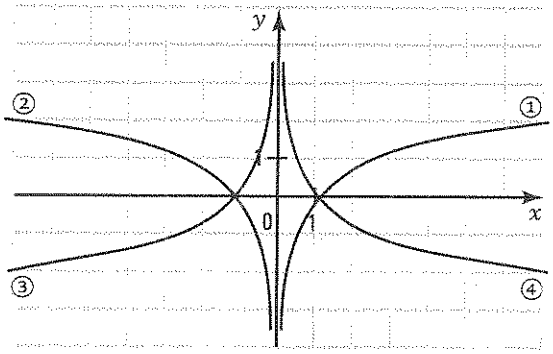
b) Graph $y = \ln x$ in the Cartesian plane.



14. a) Among the four functions represented on the right, identify

1. $y = \ln x$ ① 2. $y = -\ln x$ ④

3. $y = \ln(-x)$ ② 4. $y = -\ln(-x)$ ③

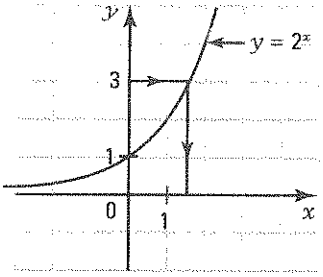


b) Complete the following table.

	$f(x) = \ln x$	$f(x) = -\ln x$	$f(x) = \ln(-x)$	$f(x) = -\ln(-x)$
Domain	$\mathbb{R}_+^*$	$\mathbb{R}_+^*$	$\mathbb{R}_-^*$	$\mathbb{R}_-^*$
Range	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
Sign	$f(x) \leq 0$ if $0 < x \leq 1$ $f(x) \geq 0$ if $x \geq 1$	$f(x) \leq 0$ if $x \geq 1$ $f(x) \geq 0$ if $0 < x \leq 1$	$f(x) \geq 0$ if $x \leq -1$ $f(x) \leq 0$ if $-1 \leq x < 0$	$f(x) \leq 0$ if $x \leq -1$ $f(x) \geq 0$ if $-1 \leq x < 0$
Variation	*Increasing*	*Decreasing*	*Decreasing*	*Increasing*

ACTIVITY 3 Exponential equation – Form $c^q = p$

The function $y = 2^x$ is represented on the right.



a) Complete the following table of values.

x	-2	-1	0	1	2	3
y	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

b) We are trying to find the exponent x such that $2^x = 3$.

1. Use the graph of the function $y = 2^x$ or the table of values to determine between which consecutive integers the exponent x is located. $1 < x < 2$
2. Use a calculator to approximate the exponent x to the nearest tenth. $x = 1.6$
3. Complete. $2^x = 3 \Leftrightarrow x$ $= \log_2 \boxed{3}$

   Calculate the exponent x using a calculator (round to the nearest hundredth). $x = 1.59$

c) Solve the equation $2^x = 5$. Round the solution to the nearest hundredth.

$2^x = 5 \Leftrightarrow x = \log_2 5$; $x = 2.32$

EXPONENTIAL EQUATION – FORM $c^q = p$

When solving an exponential equation, we use the following equivalence:

$$\boxed{c^q = p \Leftrightarrow q = \log_c p}$$

1. Isolate the power containing the unknown on one side.	$2(3)^{2x} - 10 = 0$ $2(3)^{2x} = 10$ $3^{2x} = 5$
2. Use the equivalence $c^q = p \Leftrightarrow q = \log_c p$.	$2x = \log_3 5$
3. Determine the unknown.	$x = \frac{1}{2}\log_3 5$
4. Establish the solution set.	$S = \left\{\frac{1}{2}\log_3 5\right\}$

15. Solve the following exponential equations (round solution to the nearest hundredth).

a) $3^x = 6$

$x = \log_3 6 \approx 1.63$

b) $5^{2x} = 10$

$x = \frac{1}{2}\log_5 10 \approx 0.72$

c) $2^x + 1 = 4$

$x = \log_2 3 \approx 1.58$

d) $(0{,}5)^x = 3$

$x = \log_{0.5} 3 \approx -1.58$

e) $2^{2x} = 7$

$x = \frac{1}{2}\log_2 7 \approx 1.40$

f) $2(10)^x - 3 = 5$

$x = \log 4 \approx 0.60$

16. Determine, if it exists, the zero of the following exponential functions.

a) $y = -2(3)^{x-1} + 1$ $\quad$ $1 + \log_3(0.5)$

b) $y = 3\left(\frac{1}{2}\right)^{x+2} - 10$ $\quad$ $-2 + \log_{\frac{1}{2}}\left(\frac{10}{3}\right)$

c) $y = -2\left(\frac{2}{3}\right)^{2(x+1)} - 1$ $\quad$ No zero.

17. Given $f(x) = 2(5)^x - 1$. Determine, if possible, the value of x for which

a) $f(x) = 2$ $x = \log_5 1.5 \approx 0.25$ b) $f(x) = 0$ $x = \log_5 0.5 \approx -0.43$ c) $f(x) = -2$ Impossible

18. The bacterial growth within a culture is given by the rule $y = 10(2)^x$ where y represents the number of bacteria and x represents the elapsed time, in hours, since the beginning of the experiment.

a) What rule enables you to calculate the elapsed time as a function of a given number of bacteria in the culture? $x = \log_2(0.1y)$

b) Determine

1. the number of bacteria in the culture after 5 hours. 320 bacteria
2. the elapsed time since the beginning of the experiment if we observe 1000 bacteria. $x = \log_2 100 \approx 6.64$ h $\approx$ 6 h and 38 min.

19. The accumulated value $c(t)$ after t years of an initial capital c_0 invested at an interest rate i compounded n times per year is calculated according to the rule

$$c(t) = c_0\left(1 + \frac{i}{n}\right)^{nt}$$

a) What is the accumulated value of a \$1000 capital invested at an interest rate of 10% after 7 years if the interest is compounded

1. annually. \$1948.72
2. every 3 months. \$1996.50
3. every month. \$2007.92
4. every day. \$2013.56

b) After how many years will the accumulated value of a \$1000 capital, invested at an interest rate of 8%, be worth \$2000 if the interest is compounded

1. annually. $\log_{1.08} 2 = 9$ yrs
2. every 3 months. $\frac{1}{2}\log_{1.02} 2 = 8.75$ yrs

20. One litre of water evaporates by losing 1% of its volume each hour. After how long will the volume be 950 ml? $950 = 1000(0.99)^t \Leftrightarrow t = \log_{0.99} 0.95 \approx 5.10$ h $\approx$ 5 h and 6 min

21. A grocery basket for 4 people cost \$150 in the year 2000. The inflation rate remained at 3% for the following years.

a) How much did this same basket cost in 2005? \$173.89

b) In what year will this basket cost \$185? $\log_{1.03}\left(\frac{185}{150}\right) = 7.09$. In 2007.

22. A village of 1000 inhabitants increases at a rate of 10% per year. A neighbouring village of 2000 inhabitants decreases at a rate of 5% per year. After how many years will these two villages have the same population?

$1000(1.1)^t = 2000(0.95)^t$; $t = 4.73$ yrs. After 4 years and approximately 9 months.

23. A car loses 15% of its value for the first 3 years and 10% of its value for the following years. After how many years will a car purchased for \$30 000 be worth \$10 778? After 8 years.

4.4 Logarithmic function

ACTIVITY 1 Logarithmic function $f(x) = a \log_c b(x - h) + k$

The basic logarithmic function $y = \log_c x$ can be transformed into a logarithmic function defined by the rule $f(x) = a \log_c b(x - h) + k$.

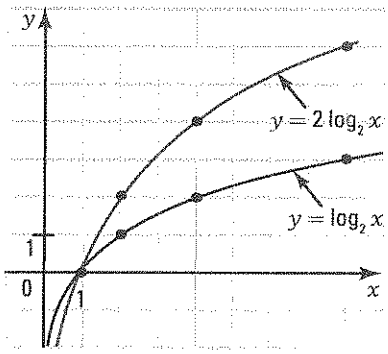
Consider the logarithmic function $y = \log_2 x$ and the logarithmic function $y = 2 \log_2 2(x - 3) - 2$.

a) Identify the parameters a, b, h and k. ***a = 2, b = 2, h = 3, k = −2***

b) The introduction of the parameters a, b, h and k into the rule of the basic logarithmic function causes a series of transformations on the graph. Complete the description of this process.

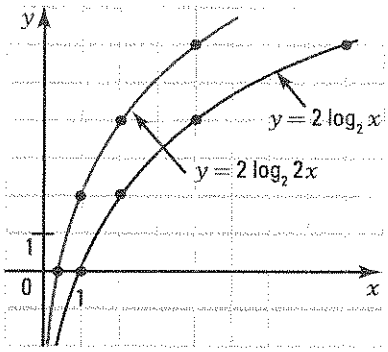
1. From the graph of $y = \log_2 x$, we obtain the graph of $y = 2 \log_2 x$ by the transformation $(x, y) \rightarrow$ ***(x, 2y)***.
   The parameter a causes ***a vertical stretch of factor 2.***

   The function $y = 2 \log_2 x$ has a vertical asymptote with equation ***x = 0***.



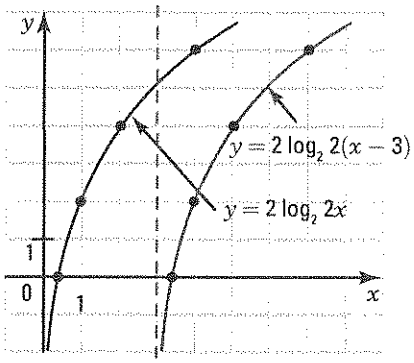
2. From the graph of $y = 2 \log_2 x$, we obtain the graph of $y = 2 \log_2 2x$ by the transformation
   $(x, y) \rightarrow$ $\left(\frac{x}{2}, y\right)$.
   The parameter b causes ***a horizontal reduction of factor 2.***

   The function $y = 2 \log_2 2x$ has a vertical asymptote with equation ***x = 0***.



3. From the graph of $y = 2 \log_2 2x$, we obtain the graph of $y = 2 \log_2 2(x - 3)$ by the transformation
   $(x, y) \rightarrow$ ***(x + 3, y)***.
   The parameter h causes ***a horizontal translation of 3 units to the right.***

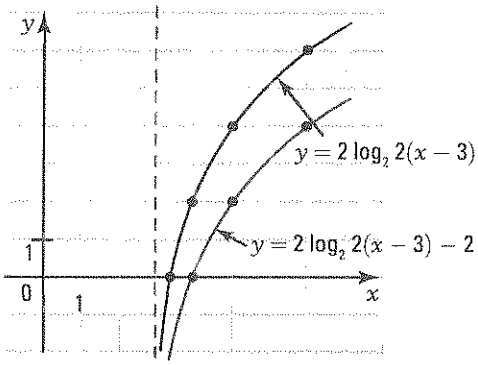
   The function $y = 2 \log_2 2(x - 3)$ has a vertical asymptote with equation ***x = 3***.



4. From the graph of $y = 2\log_2 2(x-3)$, we obtain the graph of $y = 2\log_2 2(x-3) - 2$ by the transformation $(x, y) \rightarrow$ ***(x, y − 2)***.

The parameter k causes a ***vertical translation of 2 units downward.***

The function $y = 2\log_2 2(x-3) - 2$ has a vertical asymptote with equation ***x = 3***.



c) From the tables of values, verify that the transformation $(x, y) \rightarrow \left(\frac{x}{b} + h, ay + k\right)$ directly applies the graph of $y = \log_2 x$ onto the graph of $y = 2\log_2 2(x-3) - 2$.

Indeed, (1, 0) becomes (3.5, −2), (2, 1) becomes (4, 0) and (4, 2) becomes (5, 2) by the transformation $(x, y) \rightarrow \left(\frac{x}{2} + 3, 2y - 2\right)$.

$y = \log_2 x$

x	y
1	0
2	1
4	2

$y = 2\log_2 2(x-3) - 2$

x	y
3,5	−2
4	0
5	2

LOGARITHMIC FUNCTION $f(x) = a\log_c b(x-h) + k$

- The graph of the function $f(x) = 2\log_c b(x-h) + k$ can be deduced from the graph of the basic logarithmic function $y = \log_c x$ by the transformation

$$\boxed{(x, y) \rightarrow \left(\frac{x}{b} + h, ay + k\right)}$$

- The logarithmic function $f(x) = a\log_c b(x-h) + k$ has the vertical line $x = h$ as an asymptote.
Ex.: See activity 1.

1. The following functions are defined by a rule of the form $f(x) = a\log_2 b(x-h) + k$.
$f_1(x) = 5\log_2 x$; $f_2(x) = \log_2 4x$; $f_3(x) = \log_2 (x+1)$; $f_4(x) = \log_2 x + 1$; $f_5(x) = 2\log_2 3(x-1) - 5$.

Complete the table on the right by determining the parameters a, b, h and k and by giving the rule of the transformation which enables you to graph the function from the basic function $f(x) = \log_2 x$.

	a	b	h	k	Rule
f_1	5	1	0	0	$(x, y) \rightarrow (x, 5y)$
f_2	1	4	0	0	$(x, y) \rightarrow \left(\frac{x}{4}, y\right)$
f_3	1	1	−1	0	$(x, y) \rightarrow (x - 1, y)$
f_4	1	1	0	1	$(x, y) \rightarrow (x, y + 1)$
f_5	2	3	2	−5	$(x, y) \rightarrow \left(\frac{x}{3} + 2, 2y - 5\right)$

2. In each of the following cases, a transformation is applied to the graph of the function $y = \log_5 x$. Find the rule of the function whose graph is obtained by applying the given transformation.

a) $(x, y) \rightarrow (x, -y)$ $y = -\log_5 x$

b) $(x, y) \rightarrow (2x, y)$ $y = \log_5\left(\frac{1}{2}x\right)$

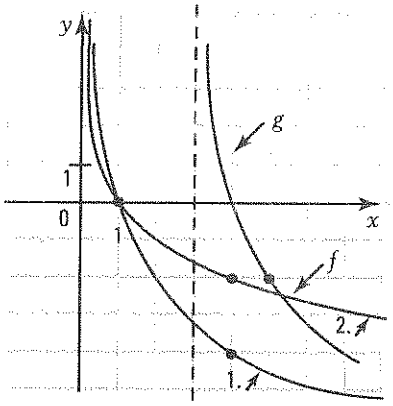
c) $(x, y) \rightarrow (x, 2y)$ $y = 2 \log_5 x$

d) $(x, y) \rightarrow (x + 2, y)$ $y = \log_5(x - 2)$

e) $(x, y) \rightarrow (x, y - 5)$ $y = \log_5 x - 5$

f) $(x, y) \rightarrow (x - 1, y + 3)$ $y = \log_5 (x + 1) + 3$

3. The function $f(x) = \log_{\frac{1}{2}} x$ is represented on the right.



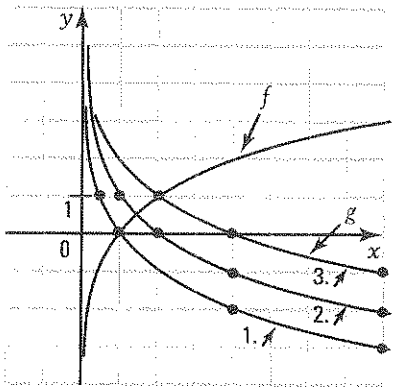
a) Explain how to obtain the graph of $g(x) = 2 \log_{\frac{1}{2}}(x - 3)$, from the graph on the right, and represent the graph of function g.

1. A vertical stretch is applied to the graph of f: (x, y) → (x, 2y).

2. A horizontal translation of 3 units to the right is then applied to the resulting graph: (x, y) → (x + 3, y).

b) What is the equation of the asymptote of function g? $x = 3$

4. The function $f(x) = \log_2 x$ is represented on the right.



a) Explain how to obtain the graph of $g(x) = -\log_2\left(\frac{1}{2}x\right) + 1$ from the graph on the right, and represent the graph of function g.

1. A reflection about the x-axis is applied to the graph of f: (x, y) → (x, −y).

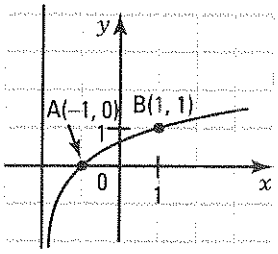
2. A horizontal stretch is applied to the resulting graph: (x, y) → (2x, y).

3. A vertical translation of 1 unit upward is then applied to the preceding graph:

b) What is the equation of the asymptote of function g? $x = 0$

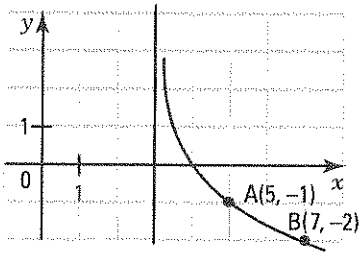
5. The functions represented below are defined by a rule of the form $y = \log_c (x - h)$. Find the rule of each function.

a)



$y = \log_3 (x + 2)$

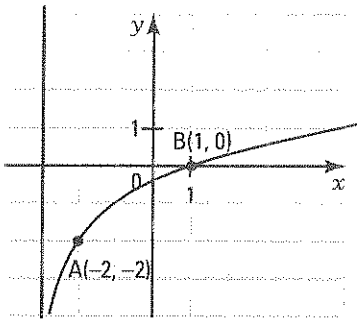
b)



$y = \log_{\frac{1}{2}}(x - 3)$

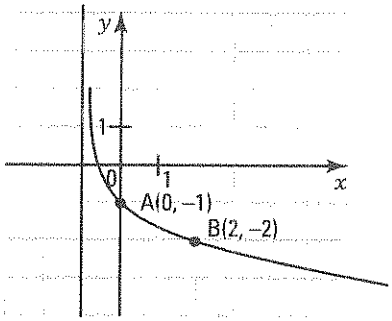
6. The functions represented below are defined by a rule of the form $y = \log_c (x - h) + k$. Find the rule of each function.

a)



$y = \log_2 (x + 3) - 2$

b)



$y = \log_{\frac{1}{3}} (x + 1) - 1$

ACTIVITY 2 Logarithmic equation – Form $\log_c p = q$

a) Consider the logarithmic equation: $5 \log_2 (2x + 1) - 3 = 12$.

1. What restriction must be placed on the variable x for $\log_2 (2x + 1)$ to exist?

   $2x + 1 > 0 \Leftrightarrow x > -\frac{1}{2}$

2. Justify the steps in solving equation.

   $5 \log_2 (2x + 1) - 3 = 12$

   $5 \log_2 (2x + 1) = 15$ — *Add 3 to each side of the equation.*

   $\log_2 (2x + 1) = 3$ — *Divide each side by 5.*

   $2x + 1 = 2^3$ — *Apply the definition of a logarithm: $\log_c p = q \Leftrightarrow p = c^q$.*

   $2x = 7$ — *Subtract 1 from each side.*

   $x = 3.5$ — *Divide each side by 2.*

3. Does the value found for x respect the restriction established in 1.? *Yes*
4. Therefore, what is the solution to the equation? *3.5*

b) A company has established that the required assembly time t for parts, in minutes, is given by $t = -20 \log_5 \left(\frac{n}{5} - 2\right) + 80$ where n represents the number of parts to be assembled.

1. What restriction must be placed on the variable n? $n > 10$
2. If an employee takes 40 minutes to assemble parts, how many parts did he assemble? (Verify if the number of parts found respects the restriction)

   $-20 \log_5 \left(\frac{n}{5} - 2\right) + 80 = 40$

   $\Leftrightarrow \log_5 \left(\frac{n}{5} - 2\right) = 2$

   $\Leftrightarrow \frac{n}{5} - 2 = 5^2$

   $\Leftrightarrow n = 135$ *(The restriction $n > 10$ is respected).*

   The employee assembled 135 parts.

LOGARITHMIC EQUATION – FORM $\log_c p = q$

When solving a logarithmic equation, we use the equivalence:

$$\boxed{\log_c p = q \Leftrightarrow p = c^q}$$

	Ex.: $3 \log_2 (2x - 4) - 5 = 4$
1. Determine the restrictions.	1. Restriction: $2x - 4 > 0 \Leftrightarrow x > 2$
2. Isolate the logarithm on one side.	2. $\log_2 (2x - 4) = 3$
3. Use the equivalence: $\log_c p = q \Leftrightarrow p = c^q$.	3. $(2x - 4) = 2^3$
4. Determine the unknown.	4. $x = 6$
5. Verify the validity of the solution.	5. The solution 6 respects the restriction.
6. Establish the solution set S.	6. $S = \{6\}$.

7. Solve the following logarithmic equations.

a) $\log_2 (x - 1) = 4$ — $S = \{17\}$

b) $2 \log_3 (x + 1) - 3 = 1$ — $S = \{8\}$

c) $\frac{1}{2}\log(5x + 10) - 1 = 0$ — $S = \{18\}$

d) $2 \log_2 (x^2 - 1) - 5 = 1$ — $S = \{-3, 3\}$

e) $5 \log_2 (x^2 + 3x - 2) - 10 = 5$ — $S = \{-5, 2\}$

f) $\log_5 (x^2 + 9) = 2$ — $S = \{-4, 4\}$

8. Find the zeros of the following logarithmic functions.

a) $f(x) = \log_5 (x + 10) - 2$ — 15

b) $f(x) = -2 \log_3 2(x + 5) - 4$ — $-\frac{1}{2}$

c) $f(x) = 2 \log_4 (x - 1) - 1$ — 3

d) $f(x) = 2 \log (x + 6) - 2$ — 4

9. The number of customers y (in thousands) willing to buy a product depending on its sale price x (in \$) is estimated by the rule $y = 20 - 4 \ln(0.1x)$.

a) Estimate the number of customers willing to buy the product when the sale price is
1. \$20. 17 227 customers 2. \$50. 13 562 customers 3. \$100. 10 790 customers

b) Estimate the sale price of the product for the following number of customers.
1. 15 000 customers \$34.90 2. 4000 customers \$545.98 3. 0 customers ~~\$1844~~ \$1484

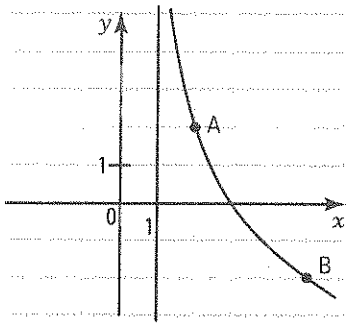
ACTIVITY 3 Sketch of the graph and study of a logarithmic function

To sketch the graph of the logarithmic function $f(x) = a \log_c b(x - h) + k$,

1. we draw the vertical asymptote with equation $x = h$.
2. we determine the domain of the function.
3. we locate, using the rule, 2 points on the logarithmic curve.
4. we sketch the logarithmic curve.

a) Sketch the graph of the following functions.

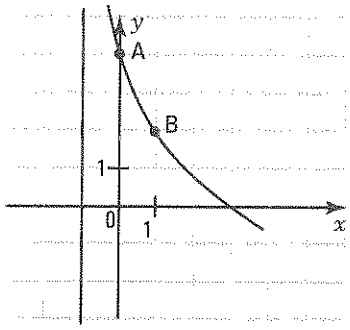
1. $f_1(x) = 2\log_{\frac{1}{2}} 2(x-1) + 4$



$domf_1 =]1, +\infty[$

$A(2, 2); B(5, -2)$

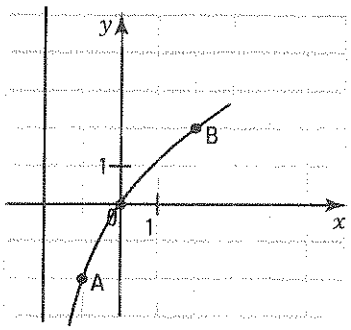
2. $f_2(x) = -2\log_2 (x+1) + 4$



$domf_2 =]-1, +\infty[$

$A(0, 4); B(1, 2)$

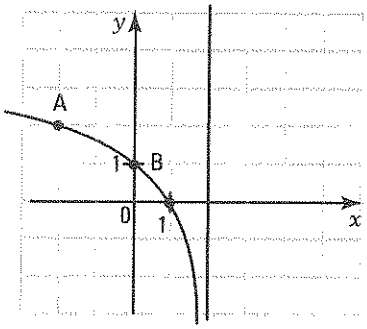
3. $f_3(x) = 2\log_2 2(x+2) - 4$



$domf_3 =]-2, +\infty[$

$A(-1, -2); B(2, 4)$

4. $f_4(x) = -\log_{\frac{1}{2}} -\frac{1}{2}(x-2) + 1$



$domf_4 =]-\infty, 2[$

$A(-2, 2); B(0, 1)$

b) Find the zero of each of the preceding functions.

1. f_1 : 3 2. f_2 : 3 3. f_3 : 0 4. f_4 : 1

c) Do a study of each of the preceding functions and complete the following table.

	a	b	h	k	Asymptote	Domain	Range	Zero	Sign	Variation
f_1	2	2	1	4	$x = 1$	$]1, +\infty[$	$\mathbb{R}$	3	$f_1(x) \geq 0$ if $x \in]1, 3]$ $f_1(x) \leq 0$ if $x \in [3, +\infty[$	Decreasing
f_2	−2	1	−1	4	$x = -1$	$]-1, +\infty[$	$\mathbb{R}$	3	$f_2(x) \geq 0$ if $x \in]-1, 3]$ $f_2(x) \leq 0$ if $x \in [3, +\infty[$	Decreasing
f_3	2	2	−2	−4	$x = -2$	$]-2, +\infty[$	$\mathbb{R}$	0	$f_3(x) \leq 0$ if $x \in]-2, 0]$ $f_3(x) \geq 0$ if $x \in [0, +\infty[$	Increasing
f_4	−1	$-\frac{1}{2}$	2	1	$x = 2$	$]-\infty, 2[$	$\mathbb{R}$	1	$f_4(x) \geq 0$ if $x \in]-\infty, 1]$ $f_4(x) \leq 0$ if $x \in [1, 2[$	Decreasing

STUDY OF A LOGARITHMIC FUNCTION

- To sketch the graph of a logarithmic function,
  1. we draw the vertical asymptote.
  2. we determine the domain of the function.
  3. we locate, using the rule, 2 points on the logarithmic curve.
  4. we sketch the logarithmic curve.
- We do a study of a logarithmic function using the sketch of the graph.
  - dom $f =]h, +\infty[$ if $b > 0$ and dom $f =]-\infty, h[$ if $b < 0$.
  - ran $f = \mathbb{R}$.
  - The sign of f is determined from the graph.
  - The variation of f is determined from the graph.
  - The function f has no extrema.

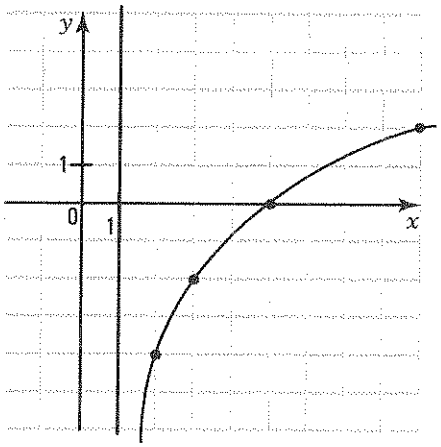
  Ex.: See activity 3.

10. For each of the following logarithmic functions, determine

1. the equation of the asymptote.
2. the domain.
3. a sketch of the graph.
4. the range.
5. the zero.
6. the sign.
7. the variation.

Rule	1	2	3	4	5	6	7
$f_1(x) = -3 \log_4 2(x+1) + 3$	$x = -1$	$]-1, +\infty[$	$(1, 0)$; $(7, -3)$	$\mathbb{R}$	1	$f_1(x) \geq 0$ if $x \in]-1, 1]$ $f_2(x) \leq 0$ if $x \in [1, +\infty[$	Decreasing
$f_2(x) = 2 \log_2 2(x+3)$	$x = -3$	$]-3, +\infty[$	$(-1, 4)$; $(1, 6)$	$\mathbb{R}$	$-\frac{5}{2}$	$f_2(x) \leq 0$ if $x \in]-3, -\frac{5}{2}]$ $f_2(x) \geq 0$ if $x \in [-\frac{5}{2}, +\infty[$	Increasing
$f_3(x) = \log_3(-x+1) - 2$	$x = 1$	$]-\infty, 1[$	$(0, -2)$; $(-2, -1)$	$\mathbb{R}$	-8	$f_3(x) \geq 0$ if $x \in]-\infty, -8]$ $f_3(x) \leq 0$ if $x \in [-8, 1[$	Decreasing
$f_4(x) = -\log_2(-x+3) + 4$	$x = 3$	$]-\infty, 3[$	$(-1, 2)$; $(1, 3)$	$\mathbb{R}$	-13	$f_4(x) \leq 0$ if $x \in]-\infty, -13]$ $f_5(x) \geq 0$ if $x \in [-13, 3[$	Increasing

11. Consider the function $f(x) = 2 \log_2(x-1) - 4$.



a) Determine

1. the asymptote of f. $x = 1$
2. the domain f. $]1, +\infty[$

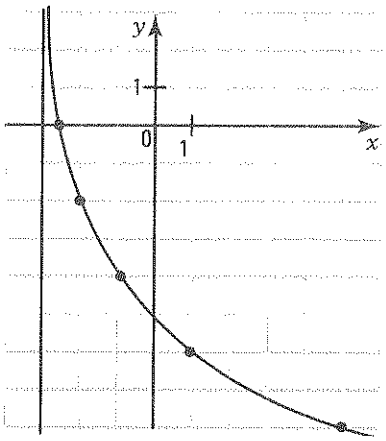
b) Represent the function f in the Cartesian plane.

x	$\frac{3}{2}$	2	3	5	9
y	-6	-4	-2	0	2

c) Determine

1. ran f. $\mathbb{R}$
2. the zero of f. 5
3. the sign of f. $f(x) \leq 0$ if $x \in]1, 5]$; $f(x) \geq 0$ if $x \in [5, +\infty[$
4. the variation of f. f is increasing over $]1, +\infty[$

12. Consider the function $f(x) = 2\log_{\frac{1}{2}}\frac{1}{2}(x+3) - 4$.



a) Determine

1. the asymptote of f. $x = -3$
2. the domain of f. $]-3, +\infty[$

b) Represent the function f in the Cartesian plane.

x	$-\frac{5}{2}$	-2	-1	1	5
y	0	-2	-4	-6	-8

c) Determine

1. ran f. $\mathbb{R}$
2. the zero of f. $-\frac{5}{2}$
3. the sign of f. $f(x) \geq 0$ if $x \in \left]-3, \frac{-5}{2}\right]$; $f(x) \leq 0$ if $x \in \left[-\frac{5}{2}, +\infty\right[$
4. the variation of f. f is decreasing over $]-3, +\infty[$.

13. Find the domain of the following logarithmic functions.

a) $y = \log_3 2(x-1) + 5$
$]1, +\infty[$

b) $y = \log_5 -2(x+3) - 5$
$]-\infty, -3[$

c) $y = \log(x^2 - 1)$
$]-\infty, -1[\cup]1, +\infty[$

d) $y = \ln(x^2 - 5x + 6)$
$]-\infty, 2[\cup]3, +\infty[$

e) $y = \log_2(x^2 + 1) + 5$
$\mathbb{R}$

f) $y = \ln(16 - x^2)$
$]-4, 4[$

14. Study the sign of the following logarithmic functions.

a) $f(x) = \log_2(x-1) - 2$
$f(x) \leq 0$ if $x \in]1, 5]$
$f(x) \geq 0$ if $x \in [5, +\infty[$

b) $f(x) = -2\log_3(x-1) + 2$
$f(x) \geq 0$ if $x \in]1, 4]$
$f(x) \leq 0$ if $x \in [4, +\infty[$

c) $f(x) = 5\log_3 2(x-1) - 10$
$f(x) \leq 0$ if $x \in \left]1, \frac{11}{2}\right]$
$f(x) \geq 0$ if $x \in \left[\frac{11}{2}, +\infty\right[$

d) $f(x) = -\frac{1}{2}\log_2 -2(x+1) + 2$
$f(x) \leq 0$ if $x \in]-\infty, -9]$
$f(x) \geq 0$ if $x \in [-9, -1[$

15. Solve the following logarithmic inequalities.

a) $2\log_3 x - 4 \geq 0$
$S = [9, +\infty[$

b) $-3\log_2 2(x-1) + 6 \geq 0$
$S =]1, 3]$

c) $2\log(x+1) - 4 \leq 0$
$S =]-1, 99]$

16. Calculate, if it exists, the initial value of the following functions.

a) $f(x) = \frac{1}{2}\log_4 2(x+8) + 1$ 2

b) $f(x) = 2\log_{\frac{1}{2}}(x-1) + 1$ Does not exist

17. Determine the variation of the following functions.

a) $f(x) = 2\log_{\frac{1}{2}} 2(x-1) + 4$
Increasing over the domain

b) $f(x) = \log_{\frac{1}{4}} -\frac{1}{2}(x+1) + 2$
Decreasing over the domain

ACTIVITY 4 Finding the rule of the inverse

a) Consider the logarithmic function $y = 3 \log_2 4(x - 1) - 6$.
Justify the steps for finding the rule of the inverse.

1. Isolate x in the equation: $y = 3 \log_2 4(x - 1) - 6$

$3 \log_2 4(x - 1) = y + 6$	***Add 6 to each side of the equation.***
$\log_2 4(x - 1) = \frac{1}{3}(y + 6)$	***Divide each side by 3.***
$4(x - 1) = 2^{\frac{1}{3}(y+6)}$	***$\log_c p = q \Leftrightarrow p = c^q$. (Definition)***
$x - 1 = \frac{1}{4}(2)^{\frac{1}{3}(y+6)}$	***Divide each side by 4.***
$x = \frac{1}{4}(2)^{\frac{1}{3}(y+6)} + 1$	***Add 1 to each side.***

2. Interchange the letters x and y to get the rule of the inverse.
The inverse of the logarithmic function: $y = 3 \log_2 4(x - 1) - 6$ is the exponential function: $y = \frac{1}{4}(2)^{\frac{1}{3}(x+6)} + 1$.

b) Consider the exponential function $y = \frac{1}{2}(2)^{\frac{1}{4}(x+3)} - 1$.
Justify the steps for finding the rule of the inverse.

1. Isolate x in the equation: $y = \frac{1}{2}(2)^{\frac{1}{4}(x+3)} - 1$

$y + 1 = \frac{1}{2}(2)^{\frac{1}{4}(x+3)}$	***Add 1 to each side of the equation.***
$2(y + 1) = 2^{\frac{1}{4}(x+3)}$	***Multiply each side by 2.***
$\frac{1}{4}(x + 3) = \log_2 2(y + 1)$	***$p = c^q \Leftrightarrow \log_c p = q$.***
$x + 3 = 4 \log_2 2(y + 1)$	***Multiply each side by 4.***
$x = 4 \log_2 2(y + 1) - 3$	***Subtract 3 from each side.***

2. Interchange the letters x and y to get the rule of the inverse.
The inverse of the exponential function: $y = \frac{1}{2}(2)^{\frac{1}{4}(x+3)} - 1$ is the logarithmic function: $y = 4 \log_2 2(x + 1) - 3$.

c) Determine the domain and range of the preceding functions and verify that

1. $\text{dom } f = \text{ran } f^{-1}$
a) dom f = ran f^{-1} =]1, +∞[***b) dom f = ran f^{-1} = ℝ***

2. $\text{ran } f = \text{dom } f^{-1}$
a) ran f = dom f^{-1} = ℝ ***b) ran f = dom f^{-1} =]–1, +∞[***

d) Complete.

1. The inverse of an exponential function in base c is a ***logarithmic function in base c.***
2. The inverse of a logarithmic function in base c is an ***exponential function in base c.***

FINDING THE RULE OF THE INVERSE

The inverse of an exponential (logarithmic) function in base c is a logarithmic (exponential) function in base c.

Ex.: See activity 4 for finding the rule.

18. For each of the following exponential functions,

1. determine the inverse.
2. verify that dom f = ranf^{-1} and that ran$f$ = dom f^{-1}.

a) $f(x) = 3^{x+1} - 6$

$f^{-1}(x) = \log_3 (x + 6) - 1$

$domf = ranf^{-1} = \mathbb{R}$

$ranf = domf^{-1} =]-6, +\infty[$

b) $f(x) = 5^{2(x-1)} - 25$

$f^{-1}(x) = \frac{1}{2} \log_5 (x + 25) + 1$

$domf = ranf^{-1} = \mathbb{R}$

$ranf = domf^{-1} =]-25, +\infty[$

c) $f(x) = -5(2)^{3(x+2)} + 10$

$f^{-1}(x) = \frac{1}{3} \log_2 \frac{-1}{5}(x - 10) - 2$

$domf = ranf^{-1} = \mathbb{R}$

$ranf = domf^{-1} =]-\infty, 10[$

19. Determine the inverse of each of the following logarithmic functions.

a) $f(x) = \log_3 (x - 1) + 2$

$f^{-1}(x) = 3^{x-2} + 1$

b) $f(x) = \log_2 -3(x + 1) - 4$

$f^{-1}(x) = \frac{-1}{3}(2)^{x+4} - 1$

c) $f(x) = \frac{1}{2} \log_5 2(x - 3) + 1$

$f^{-1}(x) = \frac{1}{2}(5)^{2(x-1)} + 3$

20. The population P (in millions of inhabitants) of a country varies according to the rule $P = 50(2)^{\frac{t}{25}}$ where t represents the number of elapsed years since the year 2000.

a) Find the rule of the function which gives the number t of years since 2000 as a function of the population P.

$t = 25 \log_2 \frac{P}{50}$

b) In what year will the population of this country reach 100 million inhabitants?

In the year 2025.

21. The time t (in years) required for a radioactive particle to disintegrate is given by: $t = 4 \log_{\frac{1}{2}}(m - 1)$ where m represents the mass (in g) of the particle at the beginning ($t = 0$).

a) Find the rule of the function which gives the mass m of the radioactive particle as a function of the elapsed time t since the beginning.

$m = \left(\frac{1}{2}\right)^{\frac{t}{4}} + 1$

b) What was the mass of the particle

8 years?

1. at the beginning? *2 g*
2. after 8 days? *1.25 g*

$t = 4 \log_{\frac{1}{2}} (m - 1)$

$\frac{t}{4} = \log_{\frac{1}{2}}(m - 1)$

$\left(\frac{1}{2}\right)^{t/4} + 1 = m$

$m(0) = \left(\frac{1}{2}\right)^{0} + 1 = 2$ g

$m\left(\frac{8}{365}\right) = \left(\frac{1}{2}\right)^{\frac{8}{1460}} + 1 = 1.996$ g.

4.5 Logarithmic calculations

ACTIVITY 1 Properties of logarithms

a) 1. Complete the equivalence: $\log_a x = y \Leftrightarrow x =$ a^y

2. Use the preceding equivalence to explain why

1) $\log_a 1 = 0$ since $a^0 = 1$

2) $\log_a a = 1$ since $a^1 = a$

3) $\log_a a^n = n$ since $a^n = a^n$

b) We know, by the definition of a logarithm, that the logarithm in base a of a positive number M is the exponent which must be given to the base a to get M. Therefore, deduce $a^{\log_a M}$. M

c) Verify the preceding properties for logarithms in base 10.

1. $\log_{10} 1 = 0$ 2. $\log_{10} 10 = 1$ 3. $\log_{10} 10^n = n$ 4. $10^{\log_{10} M} = M$

PROPERTIES OF LOGARITHMS

- $\log_a 1 = 0$ Ex.: $\log_2 1 = 0$
- $\log_a a = 1$ Ex.: $\log_5 5 = 1$
- $\log_a a^n = n$ Ex.: $\log_5 5^2 = 2$
- $a^{\log_a M} = M$ Ex.: $10^{\log_{10} 100} = 100$

1. Calculate, using the properties of logarithms.

a) $\log_5 1 =$ 0 b) $\log_{\frac{1}{2}} \frac{1}{2} =$ 1 c) $\log_2 2^3 =$ 3

d) $\log_{\frac{1}{3}} \left(\frac{1}{3}\right)^{-2} =$ -2 e) $\log_{\frac{2}{3}} 1 =$ 0 f) $2^{\log_2 3} =$ 3

g) $\left(\frac{3}{2}\right)^{\log_{\frac{3}{2}} 5} =$ 5 h) $\log_4 4^{\frac{3}{2}} =$ $\frac{3}{2}$ i) $\log_{\frac{4}{9}} \left(\frac{4}{9}\right)^{\frac{1}{2}} =$ $\frac{1}{2}$

2. Calculate the following real numbers.

a) $\log 10 =$ 1 b) $\ln e^2 =$ 2 c) $\log 10^{-3} =$ -3

d) $10^{\log 100} =$ 100 e) $e^{\ln 2} =$ 2 f) $10^{\log 10} =$ 10

g) $e^{\ln e^2} =$ e^2 h) $10^{\log 10^{-1}} =$ 10^{-1} i) $\ln e^e =$ e

3. Use the identity $M = a^{\log_a M}$ to write the real number 5 as a power of

a) 2 $2^{\log_2 5}$ b) 10 $10^{\log 5}$ c) e $e^{\ln 5}$ d) $\frac{3}{2}$ $\left(\frac{3}{2}\right)^{\log_{\frac{3}{2}} 5}$

ACTIVITY 2 Properties of logarithms (continued)

a) Verify that

1. $\log_2 (8 \times 4) = \log_2 8 + \log_2 4$ $\log_2 (8 \times 4) = \log_2 32 = 5$; $\log_2 8 + \log_2 4 = 3 + 2 = 5$.
2. $\log_2 \left(\frac{8}{4}\right) = \log_2 8 - \log_2 4$ $\log_2 \left(\frac{8}{4}\right) = \log_2 2 = 1$; $\log_2 8 - \log_2 4 = 3 - 2 = 1$.
3. $\log_2 2^3 = 3 \log_2 2$ $\log_2 2^3 = \log_2 8 = 3$; $3 \log_2 2 = 3 \times 1 = 3$.

b) Justify the steps showing that $\log_a(xy) = \log_a x + \log_a y$.

	Steps	Justifications
1	$a^{\log_a (xy)} = a^{\log_a x} \times a^{\log_a y}$	$a^{\log_a (xy)} = xy$ *and* $a^{\log_a x} \times a^{\log_a y} = xy$, $\left(\text{property: } a^{\log_a M} = M\right)$.
2	$a^{\log_a x} \times a^{\log_a y} = a^{\log_a x + \log_a y}$	$a^m \times a^n = a^{m+n}$ *(laws of exponents).*
3	$a^{\log_a (xy)} = a^{\log_a x + \log_a y}$	*Equality is transitive (see equalities 1 and 2).*
4	$\log_a (xy) = \log_a x + \log_a y$	$a^p = a^q \Rightarrow p = q$ *(see equality 3).*

c) Justify the steps showing that $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$.

	Steps	Justifications
1	$a^{\log_a \left(\frac{x}{y}\right)} = a^{\log_a x} \div a^{\log_a y}$	$a^{\log_a \left(\frac{x}{y}\right)} = \frac{x}{y}$; $a^{\log_a x} \div a^{\log_a y} = \frac{x}{y}$, $\left(\text{property: } a^{\log_a M} = M\right)$.
2	$a^{\log_a x} \div a^{\log_a y} = a^{\log_a x - \log_a y}$	$a^m \div a^n = a^{m-n}$ *(laws of exponents).*
3	$a^{\log_a \left(\frac{x}{y}\right)} = a^{\log_a x - \log_a y}$	*Equality is transitive (see equalities 1 and 2).*
4	$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$	$a^p = a^q \Rightarrow p = q$ *(see equality 3).*

d) Justify the steps showing that $\log_a x^n = n \log_a x$.

	Steps	Justifications
1	$a^{\log_a x^n} = (a^{\log_a x})^n$	$a^{\log_a (x^n)} = x^n$; $(a^{\log_a x})^n = x^n$, $\left(\text{property: } a^{\log_a M} = M\right)$.
2	$a^{\log_a x^n} = a^{n \log_a x}$	$(a^p)^q \times a^{pq}$ *(laws of exponents).*
3	$a^{\log_a x^n} = a^{n \log_a x}$	*Equality is transitive (see equalities 1 and 2).*
4	$a^{\log_a x^n} = n \log_a x$	$a^p = a^q \Rightarrow p = q$ *(see equality 3).*

PROPERTIES OF LOGARITHMS (continued)

- Logarithm of a product.

$\boxed{\log_a xy = \log_a x + \log_a y}$ Ex.: $\log(100 \times 10) = \log 100 + \log 10$

- Logarithm of a quotient.

$\boxed{\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y}$ Ex.: $\log\left(\frac{100}{10}\right) = \log 100 - \log 10$

- Logarithm of a power.

$\boxed{\log_a x^n = n \log_a x}$ Ex.: $\log 10^2 = 2 \log 10$

4. a) True or false?

1. $\log_a x^n = (\log_a x)^n$ ***False***
2. $\log_a x^n = n \log_a x$ ***True***

b) Calculate

1. $(\log 10)^2$ $= 1^2 = 1$
2. $\log 10^2$ $= 2 \log 10 = 2$

5. a) Show that $\log_a \sqrt[n]{x} = \frac{1}{n}\log_a x$. $\log_a \sqrt[n]{x} = \log_a x^{\frac{1}{n}} = \frac{1}{n}\log_a x$

b) Simplify.

1. $\log_a \sqrt[3]{a}$ $\frac{1}{3}$
2. $\log_a \sqrt[n]{a^m}$ $\frac{m}{n}$

6. Knowing that $\log_2 3 \approx 1.5850$ and that $\log_2 5 \approx 2.3219$, determine

a) $\log_2 15$ ~~3.6802~~ 3.9069 b) $\log_2\left(\frac{5}{3}\right)$ ***0.7369*** c) $\log_2\left(\frac{3}{5}\right)$ ***−0.7369***

d) $\log_2 25$ ***4.6438*** e) $\log_2 \sqrt{3}$ ***0.7925*** f) $\log_2 \sqrt[3]{9}$ ***1.0567***

7. Use the properties of logarithms to express the following logarithms as a sum or difference of logarithms.

a) $\log_2(3 \times 5)$ $\log_2 3 + \log_2 5$ b) $\log_2(3 \times 5 \times 7)$ $\log_2 3 + \log_2 5 + \log_2 7$

c) $\log_2\left(\frac{5}{7}\right)$ $\log_2 5 - \log_2 7$ d) $\log_2\left(\frac{2\times5}{7}\right)$ $\log_2 2 + \log_2 5 - \log_2 7$

e) $\log_2\left(\frac{2\times5}{3\times7}\right)$ $\log_2 2 + \log_2 5 - \log_2 3 - \log_2 7$ f) $\log_2(3^2 \times 5^3)$ $2\log_2 3 + 3\log_2 5$

g) $\log_2\left(\frac{3^2}{5^3}\right)$ $2\log_2 3 - 3\log_2 5$ h) $\log_2\left(\frac{3^2\times7^3}{5^2}\right)$ $2\log_2 3 + 3\log_2 7 - 2\log_2 5$

8. Write the following expressions as a single logarithm.

a) $\log_2 3 + \log_2 5$ $\log_2 15$ b) $2\log_2 3 + \log_2 5$ $\log_2 45$

c) $\log_3 7 - \log_3 2$ $\log_3 \frac{7}{2}$ d) $\log_5 2 + \log_5 3 - \log_5 7$ $\log_5 \frac{6}{7}$

e) $\log_2 7 + \log_2 5 - \log_2 6 - \log_2 3$ $\log_2 \frac{35}{18}$ f) $3\log_5 2 + \log_5 3 - 2\log_5 7$ $\log_5 \frac{24}{49}$

9. True or false?

a) $\log_2 8 + \log_2 16 = 7$ ___T___

b) $\log_3 25 = 2 \log_3 5$ ___T___

c) $\log_2 (2 + 4) = \log_2 2 + \log_2 4$ ___F___

d) $\log 2 \times \log 5 = \log (2 + 5)$ ___F___

e) $\log_2 \left(\frac{16}{8}\right) = \frac{\log_2 16}{\log_2 8}$ ___F___

f) $\log_3 12 - \log_3 6 = \log_3 2$ ___T___

g) $\log_2 (5^2)^3 = 2 \log_2 5^3$ ___T___

h) $\log_c \left(\frac{1}{c}\right)^n = -n$ ___T___

10. Write x in terms of a, b and c.

a) $\ln x = \ln a + \ln b - \ln c$ ___$\frac{ab}{c}$___

b) $\ln x = 2 \ln a + 3 \ln b - \frac{1}{2} \ln c$ ___$\frac{a^2b^3}{\sqrt{c}}$___

11. Knowing that $x = \log_a 2$, $y = \log_a 3$ and $z = \log_a 5$, express the following logarithms in terms of x, y and z.

a) $\log_a 30$ ___$x + y + z$___

b) $\log_a \frac{6}{5}$ ___$x + y - z$___

c) $\log_a \frac{2}{15}$ ___$x - y - z$___

d) $\log_a 12$ ___$2x + y$___

e) $\log_a 360$ ___$3x + 2y + z$___

f) $\log_a \frac{72}{25}$ ___$3x + 2y - 2z$___

12. Write the following expressions as a single logarithm.

a) $4 \ln 2x + \ln \left(\frac{6}{x}\right) - 2 \ln 2x =$ ___$\ln 16x^4 + \ln \frac{6}{x} - \ln 4x^2 = \ln 24x$___

b) $3 \log_2 a - 2 \log_2 b + 3 =$ ___$\log_2 a^3 - \log_2 b^2 + \log_2 8 = \log_2 \left(\frac{8a^3}{b^2}\right)$___

c) $2(\log x^2 - \log 2x) + 3 \log x =$ ___$2\left(\log \frac{x^2}{2x}\right) + \log x^3 = \log \frac{x^5}{4}$___

d) $2 \log x - 3 \log y + \frac{1}{2} \log 4 =$ ___$\log x^2 - \log y^3 + \log 2 = \log \frac{2x^2}{y^3}$___

13. Write the following expressions as a single logarithm.

a) $\log_2 (x + 3) + \log_2 (x - 3)$ ___$\log_2 (x^2 - 9)$___

b) $\log_2 (x + 3) - \log_2 (x - 2)$ ___$\log_2 \left(\frac{x+3}{x-2}\right)$___

c) $3 \log (x - 1) - 2 \log x$ ___$\log \left(\frac{(x-1)^3}{x^2}\right)$___

d) $\log (x^2 - 1) - \log (x - 1)$ ___$\log (x + 1)$; $x \neq 1$___

e) $\ln (x^2 + 5x + 6) - \ln (x + 3)$ ___$\ln (x + 2)$; $x \neq -3$___

f) $\ln (2x^2 - 5x - 3) - \ln (x^2 - 9)$ ___$\ln \left(\frac{2x+1}{x+3}\right)$; $x \neq 3$___

14. Calculate the numerical value of the following expressions.

a) $\log_{\frac{1}{2}} 8 + 3 \log_2 4^2 - 10 \log_3 \sqrt{3} + 2 \log_5 1$ ___$= -3 + 12 - 5 + 0 = 4$___

b) $8 \log_4 2 - 4 \log_2 \sqrt{8} - 2 \log_{\frac{1}{2}} 8 - \log_2 4^2$ ___$= 4 - 6 + 6 - 4 = 0$___

c) $3 \log_a a^2 + 2 \log_a \left(\frac{1}{a}\right) - 5 \log_a 1$ ___$= 6 - 2 - 0 = 4$___

15. Knowing that $\log_a 2 = 3$ and $\log_a 3 = 5$, calculate $\log_a 6a^3$.

$\log_a 6a^3 = \log_a 2 + \log_a 3 + 3 \log_a a = 11$

16. Knowing that $\log_a x = 2$ and $\log_a y = 6$, calculate $\log_a \left(\frac{ax^3}{\sqrt{y}}\right)$.

$\log_a \left(\frac{ax^3}{\sqrt{y}}\right) = \log_a a + 3 \log_a x - \frac{1}{2} \log_a y = 4$

17. Knowing that $\log_a 2 = x$ and $\log_a 3 = y$, express the following in terms of x and y.

a) $\log_a 18a$ $\quad \log_a (2 \times 3^2 \times a) = \log_a 2 + 2 \log_a 3 + \log_a a = x + 2y + 1$

b) $\log_a \frac{12a^2}{\sqrt[3]{6}}$ $\quad \log_a \left(\frac{2^2 \times 3 \times a^2}{2^{\frac{1}{3}} 3^{\frac{1}{3}}}\right) = \log_a \left(2^{\frac{5}{3}} \times 3^{\frac{2}{3}} \times a^2\right) = \frac{5}{3}x + \frac{2}{3}y + 2$

c) $\log_a \left(\frac{27}{4a^2}\right)$ $\quad \log_a \left(\frac{3^3}{2^2 a^2}\right) = 3y - 2x - 2$

18. If $\log_a x = 2$, $\log_a y = 3$ and $\log_a z = 4$, find the numerical value of $\log_a \left(\frac{x^{-3}y^2\sqrt{z}}{a^2}\right)$.

$-3 \log_a x + 2 \log_a y + \frac{1}{2} \log_a z - 2 \log_a a = 0$

$-3(2) + 2(3) + \frac{1}{2}(4) - 2 = -6 + 6 + 2 - 2 = 0$

19. Show that: $\log_c \left(\frac{x}{y}\right) = -\log_c \left(\frac{y}{x}\right)$.

$\log_c \left(\frac{x}{y}\right) = \log_c \left(\frac{y}{x}\right)^{-1} = -1 \log_c \frac{y}{x}$

20. Show that: $\log_{\frac{1}{c}}(x) = \log_c \left(\frac{1}{x}\right)$.

Let: $y = \log_{\frac{1}{c}} (x)$. *We have:* $\left(\frac{1}{c}\right)^y = x$ *or* $c^{-y} = x$ *or* $c^y = \frac{1}{x}$ *that is to say* $y = \log_c \frac{1}{x}$.

Thus, $\log_{\frac{1}{c}} (x) = \log_c \left(\frac{1}{x}\right)$.

ACTIVITY 3 Change of base law

a) Given three positive numbers c, m and x different from 1.
Justify the steps showing that: $\log_c x = \frac{\log_m x}{\log_m c}$.

	Steps	Justifications
1	$y = \log_c x \Leftrightarrow c^y = x$	*Definition of a logarithm*
2	$\log_m c^y = \log_m x$	$p = q \Leftrightarrow \log_m p = \log_m q$.
3	$y \log_m c = \log_m x$	$\log_m c^y = y \log_m c$ *(property).*
4	$y = \frac{\log_m x}{\log_m c}$	*Isolate y.*
5	Thus, $\log_c x = \frac{\log_m x}{\log_m c}$	*Since* $y = \log_c x$ *(step 1).*

b) Use a calculator to verify that

1. $\log_2 8 = \frac{\log 8}{\log 2}$.

2. $\log_2 8 = \frac{\ln 8}{\ln 2}$.

CHANGE OF BASE LAW

- Given three positive numbers c, m and x different from 1, we have:

$$\boxed{\log_c x = \frac{\log_m x}{\log_m c}}$$

This law, called the **change of base law**, enables you to calculate the logarithm of a number in any base using the log and ln keys on the calculator.

Ex.: $\log_2 3 = \frac{\log 3}{\log 2} = 1.584\,96$ or $\log_2 3 = \frac{\ln 3}{\ln 2} = 1.586\,96$.

21. Use a calculator to evaluate the following (round to the nearest thousandth).

a) $\log_2 5$ 2.322 b) $\log_5 2$ ~~2.431~~ 0.431 c) $\log_7 \sqrt{2}$ ~~2.178~~ 0.178

22. Show that

a) $\log_a b \times \log_b a = 1$ $\log_a b \times \log_b a = \frac{\log b}{\log a} \times \frac{\log a}{\log b} = 1$

b) $\log_{\frac{1}{c}} x = -\log_c x$ $\log_{\frac{1}{c}} x = \frac{\log x}{\log\left(\frac{1}{c}\right)} = \frac{\log x}{\log 1 - \log c} = \frac{\log x}{-\log c} = -\frac{\log x}{\log c} = -\log_c x$

c) $\log_c\left(\frac{1}{x}\right) = \log_{\frac{1}{c}} x$ $\log_c\left(\frac{1}{x}\right) = \log_c 1 - \log_c x = -\frac{\log x}{\log c} = \frac{\log x}{\log c^{-1}} = \frac{\log x}{\log\left(\frac{1}{c}\right)} = \log_{\frac{1}{c}} x$

23. True or false?

a) $\log(a \times b) = \log a \times \log b$ F b) $\log(a + b) = \log a + \log b$ F

c) $\log\left(\frac{a}{b}\right) = \frac{\log a}{\log b}$ F d) $\log a^n = (\log a)^n$ F

e) $\log a + \log b = \log ab$ T f) $\log a - \log b = \log \frac{a}{b}$ T

g) $\log \frac{1}{a} = -\log a$ T h) $\frac{\log a}{\log b} = \log_b a$ T

24. a) Determine the real number k such that for any real number x from $\mathbb{R}_+^*$, we have: $\log x = k \ln x$.

$\log x = \frac{\ln x}{\ln 10}$ (change of base law). $k = \frac{1}{\ln 10}$

$\log x = k \ln x$

$\frac{\ln x}{\ln 10} = k \ln x$

$\frac{1}{\ln 10} = k$

b) Define the geometric transformation which enables you to obtain the graph of $y = \log x$ from the graph of $y = \ln x$.

$(x, y) \rightarrow \left(x, \frac{1}{\ln 10} y\right)$

25. Use the properties of logarithms to calculate

a) $\log_2 2^4 + \log_2 2 - \log_2 8$ 2 b) $\log_2(\log_2 2)$ 0

c) $\log_c(\log_c c^c) =$ 1 d) $\log_c\left(\frac{1}{c}\right) - \log_{\frac{1}{c}}(c)$ 0

26. Simplify.

a) $\ln e^x$ x b) $e^{\ln x}$ x c) $e^{x \ln x}$ x^x

ACTIVITY 4 Equations involving logarithms – Form $\log_c p = q$

Consider the equation $\log(x + 1) = 1 - \log(x - 2)$.

a) Determine the restrictions that the variable x must respect for both sides of the equation to be defined.

$x + 1 > 0$ and $x - 2 > 0 \Leftrightarrow x > -1$ and $x > 2$

b) Justify the steps in solving this equation.

	Steps	Justifications
	$\log(x + 1) = 1 - \log(x - 2)$	
1	$\log(x + 1) + \log(x - 2) = 1$	***Add log (x – 2) to each side.***
2	$\log(x + 1)(x - 2) = 1$	***Apply the property: log a + log b = log ab.***
3	$(x + 1)(x - 2) = 10$	***$\log_c p = q \Leftrightarrow c^q = p$.***
4	$x^2 - x - 2 = 10$	***Expand the left side.***
5	$x^2 - x - 12 = 0$	***Subtract 10 from each side.***
6	$(x + 3)(x - 4) = 0$	***Factor the non-zero side.***
7	$x + 3 = 0$ or $x - 4 = 0$	***Apply the zero product principle: ab = 0 ⇔ a = 0 or b = 0***
8	$x = -3$ or $x = 4$	***Solve each equation.***
9	We reject the solution $x = -3$.	***See restrictions in a).***
10	Thus, $S = \{4\}$	***Establish the solution set.***

EQUATIONS INVOLVING LOGARITHMS – FORM $\log_c p = q$

When solving an equation involving logarithms in the same base,

- determine the restrictions that the variable x must respect for both sides of the equation to be defined.
- use the properties of logarithms to get the form $\log_c p = q$.
- use the equivalence: $\boxed{\log_c p = q \Leftrightarrow c^q = p}$
- determine the solutions that respect the restrictions.

Ex.: See activity 4.

27. Solve the following equations.

a) $\log_6(x - 5) = 2 - \log_6 x$

Restrictions: $x > 5$ and $x > 0$

$S = \{9\}$

b) $\log(x + 3) = 1 - \log x$

Restrictions: $x > -3$ and $x > 0$

$S = \{2\}$

c) $\log_2(x + 1) + \log_2(x - 1) = 3$

Restrictions: $x > -1$ and $x > 1$

$S = \{3\}$

d) $\log_2(x + 2) = 2 - \log_2(x - 1)$

Restrictions: $x > -2$ and $x > 1$

$S = \{2\}$

ACTIVITY 5 Equation involving logarithms – Form $log_c u = log_c v$

a) Justify the steps showing the equivalence $\log_c u = \log_c v \Leftrightarrow u = v$.

$\log_c u = \log_c v \Leftrightarrow c^{\log_c u} = c^{\log_c v}$ ***$p = q \Leftrightarrow a^p = a^q$***

$\Leftrightarrow \quad u \quad = \quad v$ ***Property: $a^{\log_a M} = M$***

b) Consider the logarithmic equation $\ln(x + 2) = \ln(-x + 6) + \ln(x - 1)$.

What restrictions must the variable x respect for both sides of the equation to be defined?

$x + 2 > 0$ and $-x + 6 > 0$ and $x - 1 > 0$, that is to say $x > -2$ and $x < 6$ and $x > 1$

c) Justify the steps in solving the equation $\ln(x + 2) = \ln(-x + 6) + \ln(x - 1)$.

	Steps	Justifications
	$\ln(x + 2) = \ln(-x + 6) + \ln(x - 1)$	
1	$\ln(x + 2) = \ln(-x + 6)(x - 1)$	***$\ln a + \ln b = \ln ab$ (property).***
2	$(x + 2) = (-x + 6)(x - 1)$	***$\log_c u = \log_c v \Leftrightarrow u = v$.***
3	$x + 2 = -x^2 + 7x - 6$	***Expand the right side.***
4	$x^2 - 6x + 8 = 0$	***Write in the form $ax^2 + bx + c = 0$.***
5	$(x - 4)(x - 2) = 0$	***Factor the non-zero side.***
6	$x = 4$ or $x = 2$	***Apply the zero product principle: $ab = 0 \Leftrightarrow a = 0$ or $b = 0$.***
7	The solutions are valid. Thus, $S = \{2, 4\}$	***The solutions respect the restrictions established in b): $x > -2$ and $x < 6$ and $x > 1$.***

EQUATIONS INVOLVING LOGARITHMS – FORM $log_c u = log_c v$

When solving an equation involving logarithms in the same base,

- determine the restrictions that the variable x must respect for both sides of the equation to be defined.
- use the properties of logarithms to get the form $\log_c u = \log_c v$.
- use the equivalence: $\boxed{\log_c u = \log_c v \Leftrightarrow u = v}$
- determine the solutions that respect the restrictions.

Ex.: See activity 5.

28. Solve the following equations.

a) $\log(x + 1) = \log 6 - \log x$

Restrictions: $x > 0$ and $x > -1$

$S = \{2\}$

b) $\ln(x - 2) + \ln 3 = \ln(x + 1)$

Restrictions: $x > 2$ and $x > -1$

$S = \left\{\frac{7}{2}\right\}$

c) $\log_2(x + 1) - \log_2 2 = \log_2 5 - \log_2(x - 2)$

Restrictions: $x > -1$ and $x > 2$

$S = \{4\}$

d) $\ln(x + 3) - \ln(x + 1) = \ln(x - 3) - \ln(x - 2)$

Restrictions: $x > -3$; $x > -1$; $x > 3$; $x > 2$

$S = \varnothing$

29. Solve the following equations.

a) $\log_5 (x+2) + \log_5 (x-2) - \log_5 4 = \log_5 3$

Restrictions: $x > -2$ and $x > 2$

$S = \{4\}$

b) $\log_5 (x^2-4) - \log_5 4 = \log_5 3$

Restrictions: $x \in]-\infty, -2[\cup]2, +\infty[$

$S = \{-4, 4\}$

c) $\log_5 (x+2) = 1 - \log_5 (x-2)$

Restrictions: $x > -2$ and $x > 2$

$S = \{3\}$

d) $\log_5 (x^2-4) = 1$

Restrictions: $x \in]-\infty, -2[\cup]2, +\infty[$

$S = \{-3, 3\}$

30. a) Solve the equation $\log_2 (x+1) + \log_2 (x-1) = 3$.

Restrictions: $x > -1$ and $x > 1$; $S = \{3\}$

b) Solve the equation $\log_2 (x^2-1) = 3$.

Restrictions: $x^2 - 1 > 0 \Leftrightarrow x \in]-\infty, -1[\cup]1, +\infty[$; $S = \{-3, 3\}$

c) Explain why the equations $\log_2 (x+1) + \log_2 (x-1) = 3$ and $\log_2 (x^2-1) = 3$ are not equivalent.

The equations do not have the same solution set.

ACTIVITY 6 Exponential equation – Form $a^u = b^v$

The exponential equation $3^{x-1} = 2^{x+1}$ contains powers with different bases.
Justify the steps to find the solution.

	Steps	Justifications
	$3^{x-1} = 2^{x+1}$	
1	$\ln 3^{x-1} = \ln 2^{x+1}$	***$a = b \Leftrightarrow \ln a = \ln b$.***
2	$(x-1)\ln 3 = (x+1)\ln 2$	***$\log_a b^n = n \log_a b$ (property).***
3	$x\ln 3 - \ln 3 = x\ln 2 + \ln 2$	***Distributive property.***
4	$x(\ln 3 - \ln 2) = \ln 3 + \ln 2$	***Isolate the terms with x.***
5	$x\ln\left(\frac{3}{2}\right) = \ln 6$	***$\ln a - \ln b = \ln\left(\frac{a}{b}\right)$ and $\ln a + \ln b = \ln (ab)$.***
6	$x = \frac{\ln 6}{\ln \frac{3}{2}}$	***Divide each side by $\ln\left(\frac{3}{2}\right)$***
7	$x = \log_{\frac{3}{2}} 6$	***$\log_a u = \frac{\ln u}{\ln a}$***
	Thus, $S = \left\{\log_{\frac{3}{2}} 6\right\}$	***Use the change of base law.***

EXPONENTIAL EQUATION — FORM $a^u = b^v$

The use of logarithms enables you to solve an exponential equation involving powers of different bases.

Ex.: Solve the equation $3^{x-1} = 2^{x+1}$.

1. Use the equivalence $a = b \Leftrightarrow \ln a = \ln b$	$3^{x-1} = 2^{x+1}$ $\ln 3^{x-1} = \ln 2^{x+1}$
2. Apply the property: $\log_a b^n = n \log_a b$.	$(x-1)\ln 3 = (x+1)\ln 2$
3. Distribute and isolate the terms with x.	$x\ln 3 - \ln 3 = x\ln 2 + \ln 2$ $x(\ln 3 - \ln 2) = \ln 2 + \ln 3$
4. Apply, if necessary, the properties: $\ln a + \ln b = \ln ab$ and $\ln a - \ln b = \ln \frac{a}{b}$.	$x\ln\left(\frac{3}{2}\right) = \ln 6$
5. Determine x.	$x = \frac{\ln 6}{\ln \frac{3}{2}}$
6. Use the change of base law if necessary. $\frac{\ln a}{\ln b} = \log_b a$.	$x = \log_{\frac{3}{2}} 6$ Thus, $S = \left\{\log_{\frac{3}{2}} 6\right\}$

31. Solve the following exponential equations.

a) $3^{2x-1} = 2^{x+2}$

$S = \left\{\log_{\frac{9}{4}} 12\right\}$ $\frac{\log 12}{\log 9/4} = 1.65$

b) $2^{x+1} = 5^{1-x}$

$S = \left\{\log \frac{5}{2}\right\} = 0.398$

c) $\frac{2^{x+1}}{5^x} = 3$

$S = \left\{\log_{\frac{2}{5}} \frac{3}{2}\right\} = \frac{\log 3/2}{\log 2/5} = -0.44$

32. Solve the following equations.

a) $5^{2x} + 5^x - 6 = 0$ (hint: let $y = 5^x$)

$y^2 + y - 6 = 0$; $y = -3$ and $y = 2$; $S = \{\log_2 5\}$

b) $9^x - 5(3^x) + 6 = 0$ (hint: let $y = 3^x$)

$y^2 - 5y + 6 = 0$; $y = 2$ and $y = 3$; $S = \{1, \log_3 2\}$

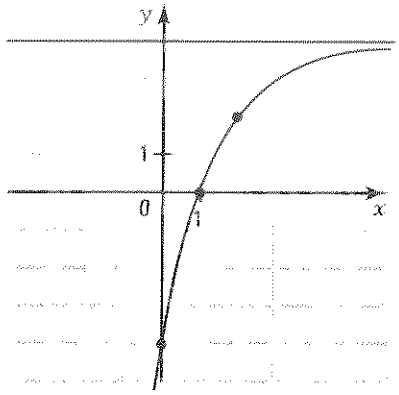
33. Julian invests \$1000 at an interest rate of 8% compounded annually. One year later, Julie also invests \$1000 at an interest rate of 10% compounded annually. After how many years, since the beginning of Julian's investment, will Julian and Julie's accumulated capital be equal? (Round your answer to the nearest tenth.)

t: number of required years. $1000(1.08)^t = 1000(1.1)^{t-1}$

After 5.2 years.

Evaluation 4

1. Consider the function $f(x) = -2\left(\frac{1}{2}\right)^{x-2} + 4$.



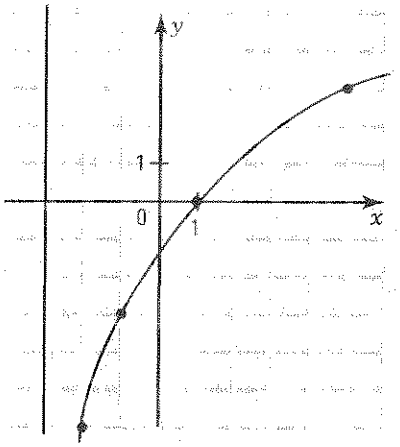
a) Represent the function f in the Cartesian plane.

x	−1	0	1	2	3	4
y	−12	−4	0	2	3	3.5

b) Determine

1. dom f. $\mathbb{R}$ 2. ran f. $]-\infty, 4[$ 3. zero of f. 1
4. sign of f. $f(x) \leq 0$ if $x \in]-\infty, 1]$; $f(x) \geq 0$ if $x \in [1, +\infty[$
5. variation of f. Increasing 6. the initial value of f. -4
7. the equation of the asymptote. $y = 4$

2. Consider the function $f(x) = 3 \log_2 \frac{1}{2}(x + 3) - 3$.



a) Represent the function f in the Cartesian plane.

x	−2.5	−2	−1	1	5	13
y	−9	−6	−3	0	3	6

b) Determine

1. dom f. $]-3, +\infty[$ 2. ran f. $\mathbb{R}$ 3. zero of f. 1
4. sign of f. $f(x) \leq 0$ if $x \in]-3, 1]$; $f(x) \geq 0$ if $x \in [1, +\infty[$
5. variation of f. Increasing
6. the initial value of f. $3 \log_2 3 - 6$
7. the equation of the asymptote. $x = -3$

3. Find the domain of the following functions.

a) $f(x) = -2\left(\frac{3}{2}\right)^{x-1} + 1$ $\mathbb{R}$

b) $f(x) = 2 \ln (2x - 1) + 3$ $\left]\frac{1}{2}, +\infty\right[$

c) $f(x) = \log_2(-2x + 6) - 1$ $]-\infty, 3[$

d) $f(x) = -5 \log_2 (x^2 - 9)$ $]-\infty, -3[\cup]3, +\infty[$

4. Solve the following equations.

a) $3(2)^x - 24 = 0$
$S = \{3\}$

b) $3(5)^x - 6 = 0$
$S = \{\log_5 2\}$

c) $2(3)^x + 1 = 0$
$S = \varnothing$

d) $3 \log_2 x - 6 = 0$
$S = \{4\}$

e) $2 \log_3 x + 4 = 0$
$S = \left\{\frac{1}{9}\right\}$

f) $3 \log_2 (x - 1) - 4 = 0$
$S = \left\{1 + 2^{\frac{4}{3}}\right\}$

5. Solve the following equations.

a) $\log_2 (x - 3) + \log_2 x = 2$

Restrictions: $x > 3$ and $x > 0$

$S = \{4\}$

b) $\ln (x - 1) + \ln (x + 6) = \ln (10 - x)$

Restrictions: $x > 1$; $x > -6$; $x < 10$

$S = \{2\}$

c) $3^{x-1} = 2^{x+1}$

$S = \left\{\log_{\frac{3}{2}} 6\right\}$

d) $2(5^{2x}) - 3(5^{2x}) + 1 = 0$

$S = \left\{\log_5 \frac{1}{2}, 0\right\}$

6. Determine the equation of the asymptote of the following functions.

a) $f(x) = 2^{3(x-1)} - 4$ *$y = -4$*

b) $f(x) = -2 \ln (2x - 6) + 1$ *$x = 3$*

7. Determine the range of the following functions.

a) $f(x) = -3(2)^{2(x+1)} + 5$ *ranf $=]-\infty, 5[$*

b) $f(x) = -2 \log (2x - 5) + 1$ *ranf $= \mathbb{R}$*

8. Determine the zero of the following functions.

a) $f(x) = 2(3)^{(x-2)} - 4$ *$\log_3 18$*

b) $f(x) = -2 \log_2 (2x + 1) + 6$ *$\frac{7}{2}$*

9. Study the sign of the following functions.

a) $f(x) = -3(2)^{(x-3)} + 12$

$f(x) \geq 0$ if $x \in]-\infty, 5]$

$f(x) \leq 0$ if $x \in [5, +\infty[$

b) $f(x) = 3 \log_{\frac{1}{2}} (x - 2) + 3$

$f(x) \leq 0$ if $x \in]2, 4]$

$f(x) \geq 0$ if $x \in [4, +\infty[$

10. Study the variation of the following functions.

a) $f(x) = -5\left(\frac{1}{2}\right)^{2x+1} + 2$ *f is increasing over $\mathbb{R}$*

b) $f(x) = -2 \log_{\frac{1}{2}} (-2x + 1) + 6$ *f is decreasing over the domain.*

11. Find the inverse of the following functions.

a) $y = -2(3)^{2(x-1)} + 4$

$y = \frac{1}{2} \log_3 \frac{-1}{2}(x - 4) + 1$

b) $y = 5 \log_2 3(x - 1) - 10$

$y = \frac{1}{3} 2^{\frac{1}{5}(x+10)} + 1$

12. Find the rule of the exponential function passing through the points A(1, –5) and B(2, –11) and having the line $y = 1$ for an asymptote.

$y = -3(2)^x + 1$

13. A logarithmic function has a rule of the form $y = \log_c (x - h) + k$. Its graph passes through the points A(4, 2) and B(10, 3) and has the line $x = 1$ for an asymptote. Find the rule of this function.

$y = \log_3 (x - 1) + 1$

14. Knowing that $x = \log_a 2$; $y = \log_a 3$ and $z = \log_a 5$, express the following logarithms in terms of x, y and z.

a) $\log_a 60$ $2x + y + z$

b) $\log_a \frac{72}{25}$ $3x + 2y - 2z$

15. Knowing that $\log_a x = 2$ and $\log_a y = 3$; calculate the numerical value of the following logarithms.

a) $\log_a x^3y^2$ 12

b) $\log_a \left(\frac{a^3x^2}{y^2}\right)$ 1

16. Write the following expressions as a single logarithm.

a) $2 \log 3 + \log 2 - \log 6 - \log 4$ $\log \frac{3}{4}$

b) $3 \ln x - 2 \ln x^2 + \ln x^3$ $\ln x^2$

c) $\log (x^2 - 1) - \log (x^2 - 3x + 2)$ $\log \frac{x+1}{x-2}$ $(x \neq 1)$

d) $\ln (2x^2 - 3x - 2) - 2 \ln (x - 2)$ $\ln \frac{2x+1}{x-2}$ $(x \neq 2)$

17. In a laboratory experiment, there are initially 12 insects. We notice that the number of insects doubles every 3 days.

a) What is the rule which gives the resulting number y of insects as a function of the number t of days since the beginning of the experiment? $y = 12(2)^{\frac{t}{3}}$

b) After how many days will there be at least 6144 insects? *27 days*

18. The formula $c(t) = c_0\left(1 + \frac{i}{n}\right)^{nt}$ gives the accumulated capital after t years of an initial capital c_0 invested at an annual interest rate i compounded n times per year.

a) A capital of \$2500 is invested at an annual interest rate of 6% compounded monthly. What will the accumulated capital be after 5 years? *\$3372.13*

b) How long will it take for a capital invested at an annual interest rate of 8% compounded twice per year to double its value? $2\,c_0 = c_0\,(1 + 0.04)^{2t}$;

$t = \log_{1.04} 2 \approx 8.84$ *years. The capital doubles after 8 years and 10 months.*

19. The population of a village decreased by approximately 2% per year since 1990. In the year 2000, the village had a population of 2500.

a) What was the population of this village in

1. 1995? $2500(0.98)^{-5} \approx 2766$ *people.* 2. 2007? $2500(0.98)^{7} \approx 2170$ *people.*

b) If the rate of decay is maintained, in what year will the population of this village be equal to half the population recorded in the year 2000? *In 2034*

20. The number of bacteria triples every thirty minutes in a culture. Four hours after the beginning of the experiment, we observe 32 805 bacteria. How many bacteria are there in the culture 1 hour and 15 minutes after the beginning of the experiment?

$y = a(3)^{2t}$ *(t in hours);* $a = 5$; $y = 5(3)^{2.5} \approx 78$ *bacteria.*

There are about 78 bacteria, 1 h 15 min after the beginning of the experiment.