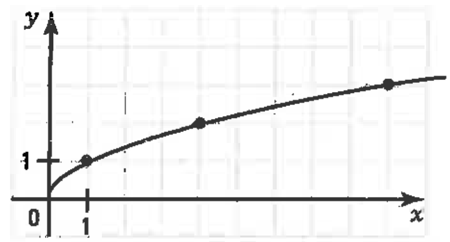
3.4 Square root function

ACTIVITY 1 Square root function

Consider the function f defined by the rule $f(x) = \sqrt{x}$.



a) What condition must be placed on x for $\sqrt{x}$ to exist in $\mathbb{R}$?
x must be positive or zero

b) Complete the following table of values.

x	0	1	4	9
y	*0*	*1*	*2*	*3*

c) Represent the function f in the Cartesian plane.

d) Determine
1. dom f = $\mathbb{R}_+$
2. ran f = $\mathbb{R}_+$
3. the zero of f. *0*
4. the initial value of f. *0*
5. the sign of f. *$f(x) \geqslant 0$ over $\mathbb{R}_+$.*
6. the variation of f. *$f \nearrow$ over $\mathbb{R}_+$.*
7. the extrema of f. *min $f = 0$*

BASIC SQUARE ROOT FUNCTION

- The function f defined by the rule:

$$\boxed{f(x) = \sqrt{x}}$$

is called the **basic square root** function.

- We have:

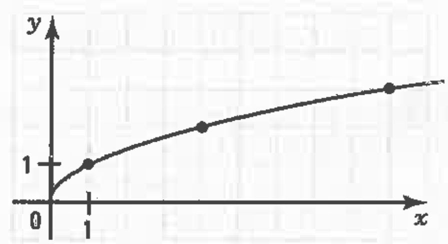
dom $f = \mathbb{R}_+$ **ran** $f = \mathbb{R}_+$

The **zero** of f is 0.   The **initial value** is 0.

Sign of f: $f(x) \geqslant 0$, $\forall\, x \in$ dom f.

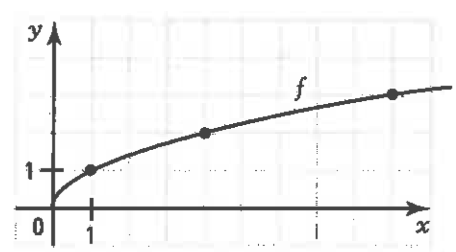
Variation of f: f is increasing, $\forall\, x \in$ dom f.

The function f has a **minimum** equal to 0.



The point O (0, 0) is the **vertex** of the function.

1. Consider the basic square root function $f(x) = \sqrt{x}$ represented on the right.
Using the graph, find the values of x for which



a) $f(x) = 3$ *$x = 9$*

b) $f(x) \geqslant 1$ *$x \in [1, +\infty[$*

c) $0 \leqslant f(x) < 2$ *$x \in [0, 4[$*

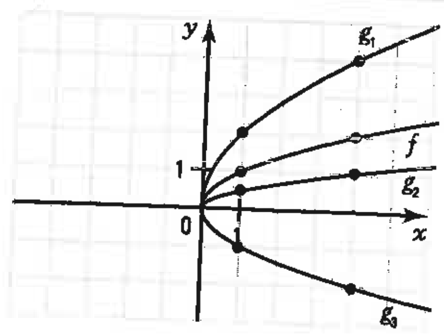
d) $f(x) < 0$ *None since the function is never strictly negative.*

e) $1 < f(x) < 3$ *$x \in]1, 9[$*

ACTIVITY 2 Square root function $f(x) = a\sqrt{b(x-h)} + k$

The basic square root function $f(x) = \sqrt{x}$ can be transformed into a square root function defined by the rule

$$\boxed{g(x) = a\sqrt{b(x-h)} + k}.$$

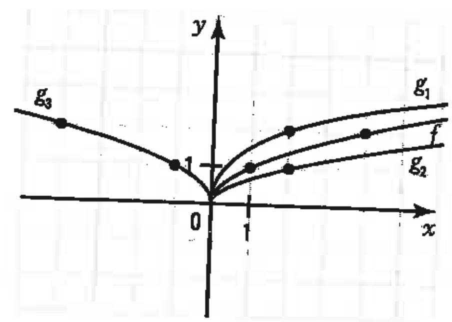


a) Consider the basic square root function $f(x) = \sqrt{x}$ and the square root function $g(x) = a\sqrt{x}$.

Represent, in the same Cartesian plane, the functions $g_1(x) = 2\sqrt{x}$, $g_2(x) = \frac{1}{2}\sqrt{x}$ and $g_3(x) = -\sqrt{x}$ and explain how to deduce the graph of g from the graph of f when

1. $a > 1$: ***by a vertical stretch.***
2. $0 < a < 1$: ***by a vertical reduction.***
3. $a = -1$: ***by a reflection about the x axis.***
4. Complete: From the graph of $f(x) = \sqrt{x}$, we obtain the graph of $g(x) = a\sqrt{x}$ by the transformation $(x, y) \rightarrow$ ***(x, ay)***.
5. Is the graph of $g(x) = a\sqrt{x}$ located in the 1st or 4th quadrant when
   1) $a > 0$? ***1st quadrant*** 2) $a < 0$? ***4th quadrant***

b) Consider the basic square root function $f(x) = \sqrt{x}$ and the square root function $g(x) = \sqrt{bx}$.

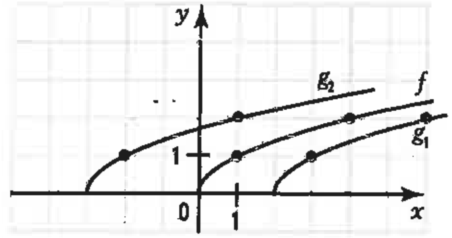


Represent, in the same Cartesian plane, the functions $g_1(x) = \sqrt{2x}$, $g_2(x) = \sqrt{\frac{1}{2}x}$ and $g_3(x) = \sqrt{-x}$ and explain how to deduce the graph of g from the graph of f when

1. $b > 1$: ***by a horizontal reduction.***
2. $0 < b < 1$: ***by a horizontal stretch.***
3. $b = -1$: ***by a reflection about the y axis.***
4. Complete: From the graph of $f(x) = \sqrt{x}$, we obtain the graph of $g(x) = \sqrt{bx}$ by the transformation $(x, y) \rightarrow$ $\left(\frac{x}{b}, y\right)$
5. In which quadrant is the graph of $g(x) = \sqrt{bx}$ when
   1) $b > 0$? ***1st quadrant*** 2) $b < 0$? ***2nd quadrant***
6. What can you say about the graph of the function $y = 2\sqrt{x}$ and that of the function $y = \sqrt{4x}$? Justify your answer.
   They are the same. In fact, $\sqrt{4x} = \sqrt{4} \cdot \sqrt{x} = 2\sqrt{x}$ (property of radicals).

c) Consider the basic square root function $f(x) = \sqrt{x}$ and the square root function $g(x) = \sqrt{x - h}$.

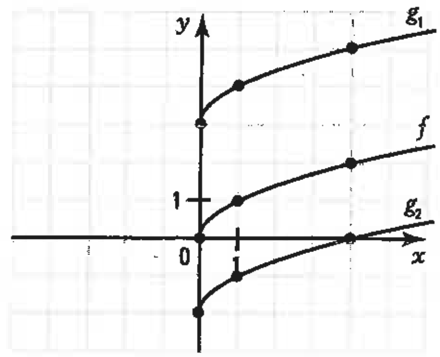
Represent, in the same Cartesian plane, the functions $g_1(x) = \sqrt{x-2}$ and $g_2(x) = \sqrt{x+3}$ and explain how to deduce the graph of g from the graph of f when



1. $h > 0$: ***by a horizontal translation to the right.***
2. $h < 0$: ***by a horizontal translation to the left.***
3. Complete: From the graph of $f(x) = \sqrt{x}$, we obtain the graph of $g(x) = \sqrt{x-h}$ by the transformation $(x, y) \rightarrow$ ***$(x + h, y)$***

d) Consider the basic square root function $f(x) = \sqrt{x}$ and the square root function $g(x) = \sqrt{x} + k$.

Represent, in the same Cartesian plane, the functions $g_1(x) = \sqrt{x}+3$ and $g_2(x) = \sqrt{x}-2$ and explain how to deduce the graph of g from the graph of f when



1. $k > 0$: ***by a vertical translation upward.***
2. $k < 0$: ***by a vertical translation downward.***
3. Complete: From the graph of $f(x) = \sqrt{x}$, we obtain the graph of $g(x) = \sqrt{x} + k$ by the transformation $(x, y) \rightarrow$ ***$(x, y + k)$***

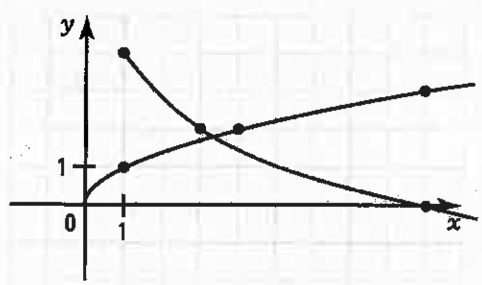
SQUARE ROOT FUNCTION $f(x) = a\sqrt{b(x - h)} + k$

The graph of the function $f(x) = a\sqrt{b(x - h)} + k$ is deduced from the graph of the basic square root function $y = \sqrt{x}$ by the transformation

$$\boxed{(x, y) \rightarrow \left(\frac{x}{b} + h, ay + k\right)}$$

Ex.: The basic square root function $y = \sqrt{x}$ and the square root function

$y = -2\sqrt{\frac{1}{2}(x - 1)} + 4$ are represented on the right.



The rule of the transformation applied to the graph of the basic square root function is:
$(x, y) \rightarrow (2x + 1, -2y + 4)$.

2. The following functions have a rule of the form $f(x) = a\sqrt{b(x-h)} + k$.
$f_1(x) = 3\sqrt{x}$, $f_2(x) = \sqrt{2x}$, $f_3(x) = \sqrt{x+4}$, $f_4(x) = \sqrt{x} + 1$ and $f_5(x) = 2\sqrt{3(x-1)} - 4$.
Complete the table on the right by determining, for each function, the parameters a, b, h and k and by giving the rule of the transformation which enables you to obtain the function from the basic function $g(x) = \sqrt{x}$.

	a	b	h	k	Rule
$f_1(x) = 3\sqrt{x}$	3	1	0	0	$(x, y) \rightarrow (x, 3y)$
$f_2(x) = \sqrt{2x}$	1	2	0	0	$(x, y) \rightarrow \left(\frac{x}{2}, y\right)$
$f_3(x) = \sqrt{x+4}$	1	1	−4	0	$(x, y) \rightarrow (x - 4, y)$
$f_4(x) = \sqrt{x} + 1$	1	1	0	1	$(x, y) \rightarrow (x, y + 1)$
$f_5(x) = 2\sqrt{3(x-1)} - 4$	2	3	1	−4	$(x, y) \rightarrow \left(\frac{x}{3} + 1, 2y - 4\right)$

3. In each of the following cases, we apply a transformation to the basic square root function $f(x) = \sqrt{x}$. Find the rule of the function obtained by applying the given transformation.

a) $(x, y) \rightarrow (-x, y)$ $y = \sqrt{-x}$

b) $(x, y) \rightarrow (x - 5, y + 2)$ $y = \sqrt{x + 5} + 2$

c) $(x, y) \rightarrow \left(\frac{x}{5}, y\right)$ $y = \sqrt{5x}$

d) $(x, y) \rightarrow (x, -4y)$ $y = -4\sqrt{x}$

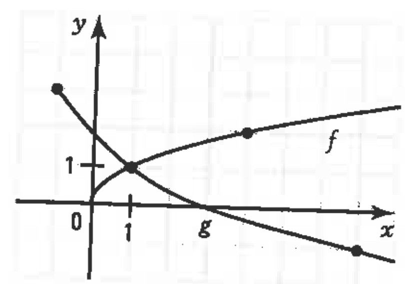
e) $(x, y) \rightarrow (2x, -6y)$ $y = -6\sqrt{\frac{1}{2}x}$

f) $(x, y) \rightarrow \left(\frac{x}{4} + 3, 3y - 5\right)$ $y = 3\sqrt{4(x - 3)} - 5$

4. Consider the functions $f(x) = \sqrt{x}$ and $g(x) = -2\sqrt{\frac{1}{2}(x+1)} + 3$.

a) Give the rule of the transformation which enables you to obtain the graph of g from the graph of f.
$(x, y) \rightarrow (2x - 1, -2y + 3)$

b) Draw the graph of g from the graph of f.

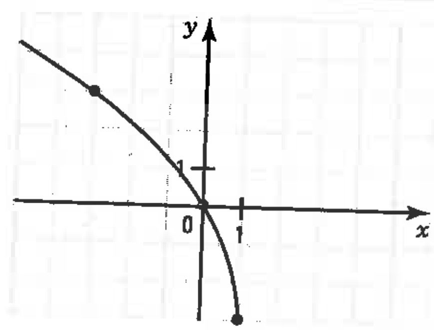


ACTIVITY 3 Graph of a square root function

Consider the function $f(x) = 6\sqrt{-\frac{1}{4}(x-1)} - 3$.

a) Identify the parameters a, b, h and k.
$a = 6$, $b = -\frac{1}{4}$, $h = 1$ and $k = -3$

b) Write the rule of the function in the form $f(x) = a\sqrt{-(x-h)} + k$.
$f(x) = 6\sqrt{-\frac{1}{4}(x-1)} - 3 = 6\sqrt{\frac{1}{4}}\sqrt{-(x-1)} - 3 = 3\sqrt{-(x-1)} - 3$



c) What are the coordinates of the vertex? $V(1, -3)$

d) Represent the function f in the Cartesian plane after completing the following table of values.

x	−3	0	1
y	3	0	−3

e) What is the zero of f? 0

ACTIVITY 4 Finding the zero of a square root function

a) Consider the function with the rule: $y = -2\sqrt{3(x+1)} + 6$.

Justify the steps in finding the zero of this function.

$-2\sqrt{3(x+1)} + 6 = 0$ *Replace y by 0.*

$\sqrt{3(x+1)} = 3$ *Isolate the square root.*

$3(x+1) = 9$ *Square each side of the equality.*

$x + 1 = 3$ *Divide each side by 3.*

$x = 2$ *Subtract 1 from each side.*

b) Under what conditions does the zero of a function $y = a\sqrt{b(x-h)} + k$ exist?

If a and k are opposite signs or if k = 0.

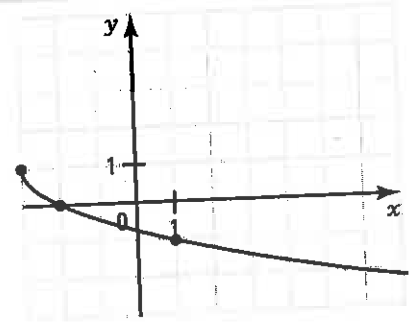
ACTIVITY 5 Study of a square root function

Consider the function f with rule $y = -\frac{1}{2}\sqrt{4x+12} + 1$.

a) Write the rule in the form $y = a\sqrt{x-h} + k$.

$y = \frac{-1}{2}\sqrt{4(x+3)} + 1 = -\sqrt{x+3} + 1$

b) Graph the function f.



c) Determine

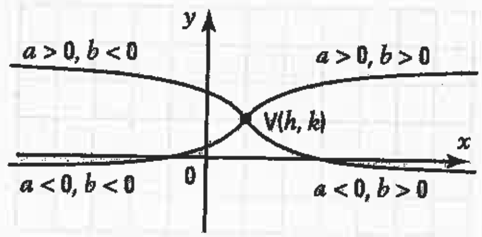
1. dom f = $[-3, +\infty[$
2. ran f = $]-\infty, 1]$
3. the zero of f (if it exists). -2
4. the initial value of f. $-\sqrt{3} + 1$
5. the sign of f. $f(x) \geq 0$ over $[-3, -2]$; $f(x) \leq 0$ over $[-2, +\infty[$
6. the variation of f. $f \nearrow$ never; $f \searrow$ over $[-3, +\infty[$
7. the extrema of f. max $f = 1$

STUDY OF A SQUARE ROOT FUNCTION

Consider the **square root** f with the rule:

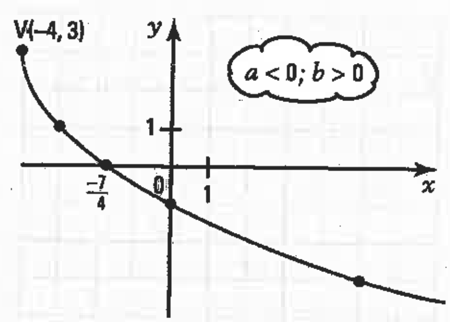
$$\boxed{f(x) = a\sqrt{b(x-h)} + k}$$

We have the following four cases:



- **dom** $f = [h, +\infty[$ if $b > 0$; **ran** $f = [k, +\infty[$ if $a > 0$;
  dom $f =]-\infty, h]$ if $b < 0$. **ran** $f =]-\infty, k]$ if $a < 0$.
- The **zero** of f exists if a and k are opposite signs or if $k = 0$.
- To study the **sign** of f,
  - we find the zero (if it exists);
  - we establish the sign of f from a sketch of the graph.
- **Variation**
  If $ab > 0$, f is increasing over the domain.
  If $ab < 0$, f is decreasing over the domain.
- **Extrema**
  If $a > 0$, f has a minimum. min $f = k$.
  If $a < 0$, f has a maximum. max $f = k$.

  Ex.: Consider the function $f(x) = -2\sqrt{x+4} + 3$ ($a = -2$, $b = 1$, $h = -4$, $k = 3$).



- Vertex: V(–4, 3)
- dom $f = [-4, +\infty[$; ran $f =]-\infty, 3]$.
- Zero: $-2\sqrt{x+4} + 3 = 0$
  $$\sqrt{x+4} = \frac{3}{2}$$
  $$x + 4 = \frac{9}{4}$$
  $$x = -\frac{7}{4}$$
- Initial value: $y = -1$
- Sign of f: $f(x) \geqslant 0$ over $\left[-4, -\frac{7}{4}\right]$; $f(x) \leqslant 0$ over $\left[-\frac{7}{4}, +\infty\right[$.
- Variation of f: f is decreasing, $\forall\, x \in$ dom f.
- max $f = 3$.

5. Write the rules of the square root functions in the form $y = a\sqrt{x-h} + k$ or $y = a\sqrt{-(x-h)} + k$.

a) $y = -2\sqrt{4x+8} + 3$

$y = -4\sqrt{x+2} + 3$

b) $y = 2\sqrt{9x-36} + 4$

$y = 6\sqrt{x-4} + 4$

c) $y = -\frac{1}{2}\sqrt{18-9x} + 1$

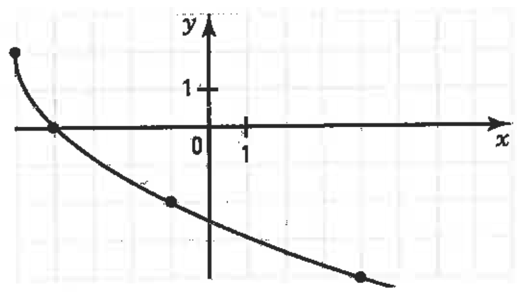
$y = -\frac{3}{2}\sqrt{-(x-2)} + 1$

d) $y = -\frac{3}{4}\sqrt{2-4x} + 7$

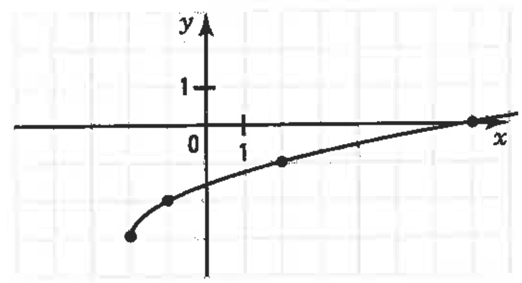
$y = -\frac{3}{2}\sqrt{-\left(x-\frac{1}{2}\right)} + 7$

6. Represent the following square root functions in the Cartesian plane.

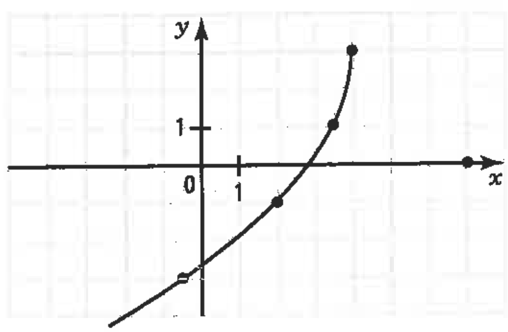
a) $y = -2\sqrt{x+5} + 2$



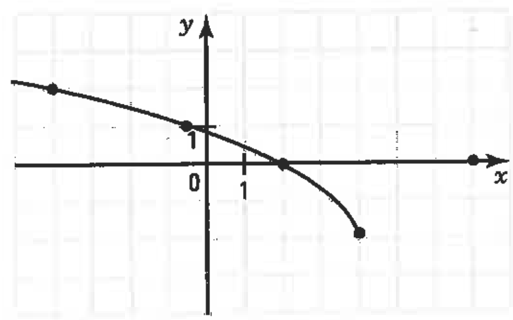
b) $y = \frac{1}{2}\sqrt{4x+8} - 3$



c) $y = -2\sqrt{-2(x-4)} + 3$

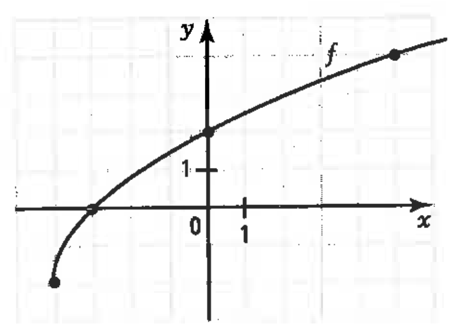


d) $y = \sqrt{-2(x-4)} - 2$



7. Consider the function $f(x) = 2\sqrt{x+4} - 2$.

a) Graph the function f.



b) Study the function f.

dom = [−4, +∞[ran = [−2, +∞[

Zero: −3; initial value: 2

$f(x) \geqslant 0$ over [−3, +∞[; $f(x) \leqslant 0$ over [−4, −3]

$f \nearrow$, $\forall x \in$ dom f

min f = −2

c) Using the graph of f, solve the inequality

1. $f(x) \geqslant 2$ *[0, +∞[*
2. $f(x) \leqslant 4$ *[−4, 5]*

8. Determine the domain and range of the following functions.

a) $y = -2\sqrt{6 - 3x} + 4$

dom =]−∞, 2]; ran =]−∞, 4]

b) $y = 3\sqrt{4x + 2} - 1$

dom = $\left[-\frac{1}{2}, +\infty\right[$; ran = [−1, +∞[

9. Determine the zero and initial value of the following functions.

a) $y = -3\sqrt{6 - 4x} + 9$

zero: $-\frac{3}{4}$, i.v.: $-3\sqrt{6} + 9$

b) $y = 2\sqrt{4x - 1} - 1$

zero: $\frac{5}{16}$, i.v.: does not exist

c) $y = 2\sqrt{x - 5} + 4$

No zero, i.v.: does not exist

d) $y = -2\sqrt{3x + 1}$

zero: $-\frac{1}{3}$, i.v.: −2

10. Consider the absolute value function $f(x) = -2|6 - 2x| + 8$ and the square root function $g(x) = 3\sqrt{\frac{1}{2}(x + 4)} - 5$. Determine

a) $g \circ f(4) =$ *1*

b) $f \circ g(-2) =$ *−12*

11. Determine the interval over which each of these functions is positive.

a) $f(x) = 3\sqrt{x + 5} - 6$

f(x) ≥ 0 over [−1, +∞[

b) $f(x) = -2\sqrt{6 + 4x} + 4$

f(x) ≥ 0 over $\left[-\frac{3}{2}, -\frac{1}{2}\right]$

c) $f(x) = \frac{1}{2}\sqrt{4 - x} + 5$

f(x) ≥ 0 over]−∞, 4]

d) $f(x) = -3\sqrt{-2x + 8} - 1$

f(x) is never positive

12. Solve the following inequalities.

a) $-2\sqrt{x + 3} + 2 \geq 0$

S = [−3, −2]

b) $\sqrt{3x + 4} < -1$

S = ∅

c) $5\sqrt{2 - x} > 4$

S = $\left]-\infty, \frac{34}{25}\right[$

d) $\sqrt{\frac{1}{2}x + 8} > 0$

S =]−16, +∞[

13. Determine the interval over which each of these functions is increasing.

a) $f(x) = 3\sqrt{-2(x - 1)} + 5$

f is never increasing

b) $f(x) = -2\sqrt{-3(x + 4)}$

f ↗ over]−∞, −4]

14. Study each of the following functions and complete the following table.

	$f_1(x) = 3\sqrt{x-2} - 1$	$f_2(x) = -2\sqrt{\frac{1}{2}(x+4)} + 6$	$f_3(x) = \sqrt{2-x} + 1$	$f_4(x) = -2\sqrt{-x} + 4$
Domain	$[2, +\infty[$	$[-4, +\infty[$	$]-\infty, 2]$	$]-\infty, 0]$
Range	$[-1, +\infty[$	$]-\infty, 6]$	$[1, +\infty[$	$]-\infty, 4]$
Zero	$\frac{19}{9}$	14	does not exist	-4
Initial value	does not exist	$-2\sqrt{2} + 6$	$\sqrt{2} + 1$	4
Sign	$f(x) \geq 0$ over $\left[\frac{19}{9}, +\infty\right[$ $f(x) < 0$ over $\left[2, \frac{19}{9}\right[$	$f(x) \geq 0$ over $[-4, 14]$ $f(x) < 0$ over $]14, +\infty[$	$f(x) \geq 0$ over $]-\infty, 2]$ $f(x) < 0$ never	$f(x) \geq 0$ over $[-4, 0]$ $f(x) < 0$ over $]-\infty, -4[$
Variation	$f \nearrow$ over $[2, +\infty[$ $f \searrow$ never	$f \nearrow$ never $f \searrow$ over $[-4, +\infty[$	$f \nearrow$ never $f \searrow$ over $]-\infty, 2]$	$f \nearrow$ over $]-\infty, 0]$ $f \searrow$ never
Extrema	min = -1	max = 6	min = 1	max = 4

ACTIVITY 6 Finding the rule of a square root function

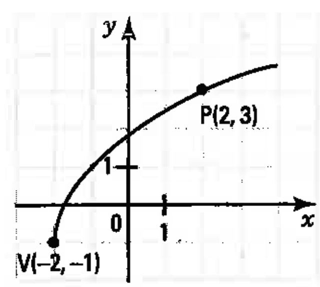
Any square root function can be written in the form $f(x) = a\sqrt{x-h} + k$ or $f(x) = a\sqrt{-(x-h)} + k$.

a) Consider the functions $f(x) = 3\sqrt{2x+4} - 5$ and $g(x) = 5\sqrt{-4x+8} - 1$.

Write the rule of each function in the form $y = a\sqrt{x-h} + k$ or $y = a\sqrt{-(x-h)} + k$.

$f(x) = 3\sqrt{2(x+2)} - 5 = 3\sqrt{2} \cdot \sqrt{x+2} - 5$ and $g(x) = 5\sqrt{-4(x-2)} - 1 = 10\sqrt{-(x-2)} - 1$

b) We consider the function represented on the right.



1. Which of the two rules $y = a\sqrt{x-h} + k$ or $y = a\sqrt{-(x-h)} + k$ corresponds to the graph of this function?

   $y = a\sqrt{-(x-h)} + k$

2. Identify h and k. $h = -2$, $k = -1$

3. Determine a knowing that the coordinates of the point P(2, 3) verify the rule of the function.

   $y = a\sqrt{x+2} - 1$

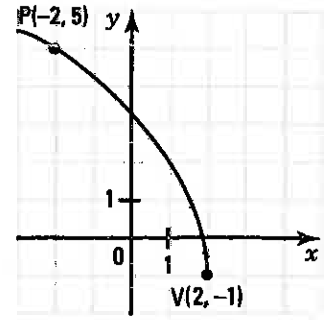
   $3 = a\sqrt{2+2} - 1$

   $4 = 2a$

   $a = 2$

4. What is the rule of the function? $y = 2\sqrt{x+2} - 1$

c) Consider the square root function whose graph has a vertex at V(2, –1) and passes through the point P(–2, 5).



1. Which of the two rules $y = a\sqrt{x-h} + k$ or $y = a\sqrt{-(x-h)} + k$ corresponds to the graph of this function?
   $y = a\sqrt{-(x-h)} + k$
2. Identify h and k. $h = 2,\ k = -1$
3. Determine a knowing that the coordinates of the point P(–2, 5) verify the rule of the function.
   $y = a\sqrt{-(x-2)} - 1;\ 5 = a\sqrt{-(-2-2)} - 1;\ 6 = 2a;\ a = 3$
4. What is the rule of the function? $y = 3\sqrt{-(x-2)} - 1$

d) What is the domain of a square root function if its rule is of the form

1. $f(x) = a\sqrt{x-h} + k$. dom $f = [h, +\infty[$
2. $f(x) = a\sqrt{-(x-h)} + k$. dom $f =]-\infty, h]$

FINDING THE RULE OF A SQUARE ROOT FUNCTION

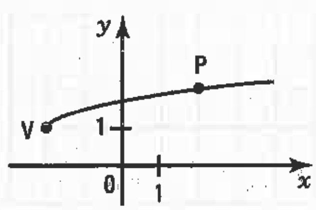
Any square root function can be written, depending on its domain, in the form:

$\boxed{f(x) = a\sqrt{x-h} + k}$ or $\boxed{f(x) = a\sqrt{-(x-h)} + k}$

The vertex V and a point P are given.

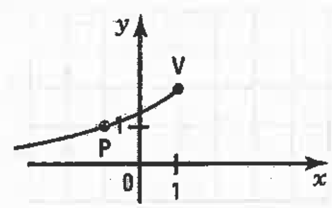
1. Determine the form of the rule, $y = a\sqrt{x-h} + k$ or $y = a\sqrt{-(x-h)} + k$.
2. Identify parameters h and k.
3. Determine a after replacing, in the rule, x and y by the coordinates of the point P.
4. Deduce the rule.

Ex.: **a)**



1. $y = a\sqrt{x-h} + k$
2. $y = a\sqrt{x+2} + 1$
3. $2 = a\sqrt{2+2} + 1$
   $1 = 2a$
   $a = \frac{1}{2}$
4. rule: $y = \frac{1}{2}\sqrt{x+2} + 1$

b)



1. $y = a\sqrt{-(x-h)} + k$
2. $y = a\sqrt{-(x-1)} + 2$
3. $1 = a\sqrt{-(-1-1)} + 2$
   $-1 = a\sqrt{2}$
   $a = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$
4. rule: $y = -\frac{\sqrt{2}}{2}\sqrt{-(x-1)} + 2$
   or $y = -\frac{1}{2}\sqrt{-2(x-1)} + 2$

15. Find the rule of each of the square root functions given its vertex V and a point P on its graph.

a) V(5, 3) and P(9, 3.5)
$y = \frac{1}{4}\sqrt{x-5} + 3$

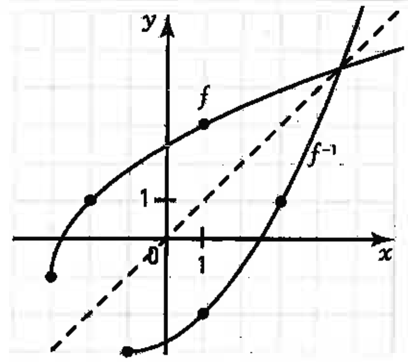
b) V(–2, –1) and P(–6, –4)
$y = -\frac{3}{2}\sqrt{-(x+2)} - 1$

c) V(–2, 4) and P(23, 2)
$y = -\frac{2}{5}\sqrt{x+2} + 4$

d) V(5, 3) and P(–13, 5)
$y = \frac{1}{3}\sqrt{-2(x-5)} + 3$

ACTIVITY 7 Inverse of a square root function

Consider the function $f(x) = 2\sqrt{x+3} - 1$.



a) In the same Cartesian plane,

1. graph the function f.
2. deduce the graph of f^{-1}.

b) Complete: The graphs of the function f and its inverse f^{-1} are symmetrical about the ***bisector of the 1st quadrant***

c) Is the inverse f^{-1} a function? Justify your answer.

Yes, because any vertical line only intersects the graph of f^{-1} in at most one point.

d) 1. Determine

1) dom f *$[-3, +\infty[$* 2) ran f *$[-1, +\infty[$* 3) dom f^{-1} *$[-1, +\infty[$* 4) ran f^{-1} *$[-3, +\infty[$*

2. Verify that

1) dom f^{-1} = ran f ***True*** 2) ran f^{-1} = dom f ***True***

e) Justify the steps in finding the rule of the inverse function f^{-1}.

1. Isolate x in the equation $y = 2\sqrt{x+3} - 1$.

$y + 1 = 2\sqrt{x+3}$ ***Add 1 to each side.***

$\frac{1}{2}(y+1) = \sqrt{x+3}$ ***Divide each side by 2.***

$\frac{1}{4}(y+1)^2 = x + 3$ ***Square both sides.***

$\frac{1}{4}(y+1)^2 - 3 = x$ ***Subtract 3 from each side.***

2. Interchange the letters x and y to obtain the rule of the inverse.
You get: $y = \frac{1}{4}(x+1)^2 - 3$.

3. What restriction must be set on the variable x? Justify your answer.

$x \geqslant -1$ since dom f^{-1} = ran f = $[-1, +\infty[$

The inverse of the square root function $y = 2\sqrt{x+3} - 1$ is therefore the function $y = \frac{1}{4}(x+1)^2 - 3 \quad (x \geqslant -1)$

The graphic representation of the inverse corresponds to a semi-parabola.

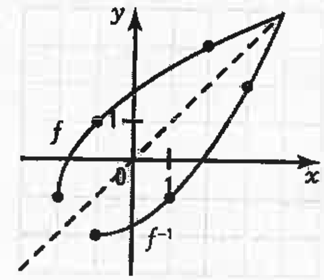
INVERSE OF A SQUARE ROOT FUNCTION

The inverse of a square root function is a function whose graph is a semi-parabola.

Ex.: $f(x) = 2\sqrt{x+2} - 1$ has the inverse:

$$f^{-1}(x) = \frac{1}{4}(x+1)^2 - 2 \qquad (x \geqslant -1)$$

Note that dom f^{-1} = ran f = $[-1, +\infty[$.



The graphs of f and f^{-1} are symmetrical about the bisector of the 1st quadrant.

16. Determine the rule of the inverse of the following functions and indicate the domain of the inverse.

a) $y = 2\sqrt{x-1} + 7$

$y = \frac{1}{4}(x-7)^2 + 1$; dom = $[7, +\infty[$

b) $y = -3\sqrt{x+4} - 1$

$y = \frac{1}{9}(x+1)^2 - 4$; dom = $]-\infty, -1]$

c) $y = 4\sqrt{-(x+3)} - 2$

$y = -\frac{1}{16}(x+2)^2 - 3$; dom = $[-2, +\infty[$

d) $y = -2\sqrt{-(x-5)} + 4$

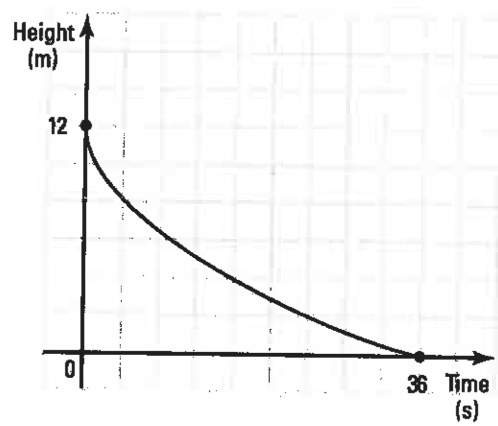
$y = -\frac{1}{4}(x-4)^2 + 5$; dom = $]-\infty, 4]$

17. At a water park, Raphael is getting ready to go down a slide.

The function f represented on the right gives Raphael's height h (in m) as a function of elapsed time t (in s) since his departure.

At what instant will he be at a height of 4 m?

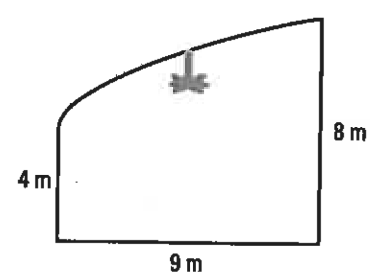
$h(t) = -2\sqrt{t} + 12$; after 16 seconds.



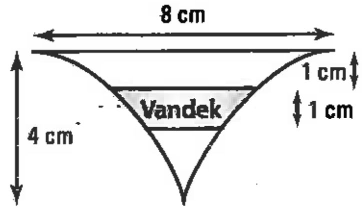
18. The lateral view of a solarium is represented by the graph on the right where the glass ceiling follows the curve of a square root function. A light is located at the centre of the room as indicated in the figure. Determine at what height the base of the light is located. (Round your answer to the nearest tenth.)

V(0, 4); $y = a\sqrt{x} + 4$; P(9, 8); $y = \frac{4}{3}\sqrt{x} + 4$.

When $x = 4.5$, the height is $y = 6.8$ m.



19. A company's logo is drawn using the graphs of two square root functions as illustrated in the figure on the right. The company's name is limited by two line segments.



a) What is the length of the upper segment?

Rule: $y = 2\sqrt{x}$. When $y = 3$, $x = 2.25$ cm. The length of the upper segment is 4.5 cm.

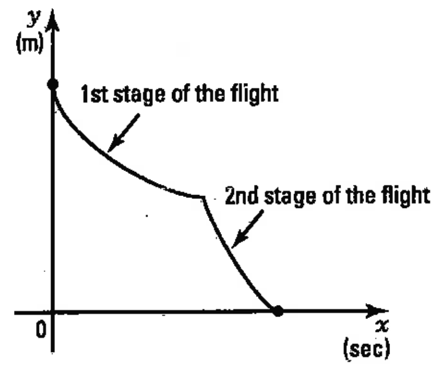
b) What is the length of the lower segment? ***2 cm***

20. The flight of a bird is observed from its takeoff at $t = 0$ from a 150 m high tower until it reaches the ground at $t = 625$ seconds.

The bird's flight is described by two square root functions represented in the figure on the right.

The 1st stage of its flight lasts 400 s and is described by the rule $y = -3\sqrt{t} + 120$ where t represents the time, in seconds, and y the height of the bird, in metres.

At the instant $t = 400$ s, the bird begins the second stage of its flight.



At what height will the bird be 500 s after the beginning of its flight?

$y = -4\sqrt{t - 400} + 60$. It will be at a height of 20 m.

3.5 Greatest integer function

ACTIVITY 1 Basic greatest integer function

a) The greatest integer of a real number x is represented by $[x]$.

1. What is the definition of the greatest integer of a real number x?

   The greatest integer of a real number x is equal to the greatest integer less than or equal to x.

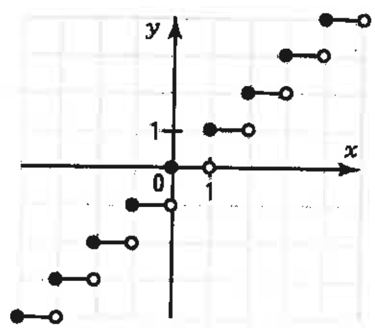
2. Calculate

   1) $[3.76]$ ***3*** 2) $[-1.25]$ ***−2***

3. In what interval is x located if

   1) $[x] = 2$ $x \in [2, 3[$ 2) $[x] = -2$ $x \in [-2, -1[$

b) 1. Graph the basic greatest integer function $f(x) = [x]$ in the Cartesian plane on the right.

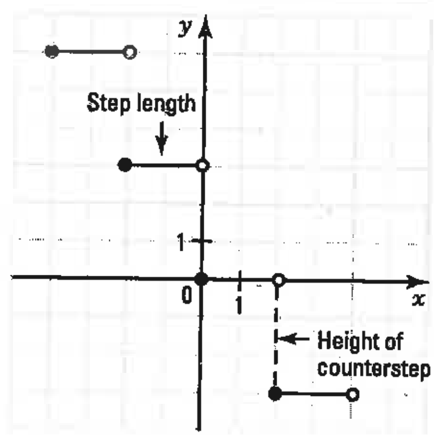


2. Determine

   1) dom f $\mathbb{R}$ 2) ran f $\mathbb{Z}$

ACTIVITY 2 General greatest integer function $f(x) = a[b(x - h)] + k$

The function $f(x) = -3\left[\frac{1}{2}(x + 4)\right] + 6$ is represented on the right.



a) Identify the parameters a, b, h and k.

   $a = -3$, $b = \frac{1}{2}$, $h = -4$, $k = 6$

b) Verify that

1. each step has a length of $\frac{1}{|b|}$. $\frac{1}{|b|} = 2$
2. The height of the counterstep is $|a|$. $|a| = |-3| = 3$

c) Determine

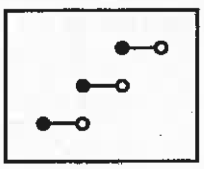
1. dom f $\mathbb{R}$ 2. ran f $\{y \mid y = -3m + 6, m \in \mathbb{Z}\}$
3. the zeros of f. $[0, 2[$ 4. the initial value of f. 0
5. the sign of f. $f(x) \geq 0$ over $]-\infty, 2[$; $f(x) \leq 0$ over $[2, +\infty[$
6. the variation of f. $f \searrow$ over $\mathbb{R}$

GREATEST INTEGER FUNCTION

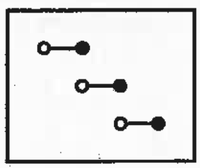
We consider the **greatest integer function** $f(x) = a[b(x - h)] + k$.

- The Cartesian graph is a step function.
- Each step has a length of $\frac{1}{|b|}$.
  - If $b > 0$, the steps are closed on the left and open on the right (●—○).
  - If $b < 0$, the steps are open on the left and closed on the right (○—●).
- The height of the counterstep is $|a|$.
- $\text{dom } f = \mathbb{R}$, $\text{ran } f = \{y \mid y = am + k,\ m \in \mathbb{Z}\}$
- – If $ab > 0$, the function is increasing.
  – If $ab < 0$, the function is decreasing.
- The function f has zeros if and only if k is a multiple of a.
- The signs of a and b help us distinguish 4 cases:

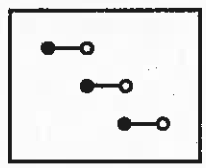
$a > 0$ and $b > 0$



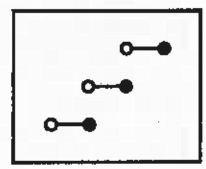
$a > 0$ and $b < 0$



$a < 0$ and $b > 0$



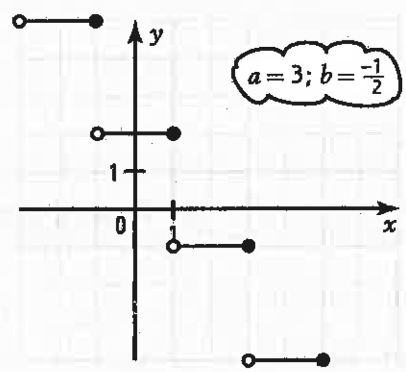
$a < 0$ and $b < 0$



Ex.: Given the function $f(x) = 3\left[-\frac{1}{2}(x - 1)\right] + 2$.

We have: $a = 3$; $b = \frac{-1}{2}$; $h = 1$ and $k = 2$.

- The length of a step is $\frac{1}{|b|} = 2$.
- The height of a counterstep is $|a| = 3$.
- $\text{dom } f = \mathbb{R}$
- $\text{ran } f = \{y \mid y = 3m + 2,\ m \in \mathbb{Z}\}$
- zeros of f: f has no zeros, since k is not a multiple of a.
- initial value of f: 2.
- $f(x) > 0$ if $x \leq 1$; $f(x) < 0$ if $x > 1$
- f is decreasing over $\mathbb{R}$ since $ab < 0$.
- f has no extrema.



1. Determine the domain and range of the following functions.

a) $y = 4\left[\frac{1}{3}(x - 5)\right] - 2$

dom = $\mathbb{R}$; ran = $\{y \mid y = 4m - 2,\ m \in \mathbb{Z}\}$

b) $y = -2[4(x + 1)] + 4$

dom = $\mathbb{R}$; ran = $\{y \mid y = -2m + 4,\ m \in \mathbb{Z}\}$

2. Determine the zeros of the following functions.

a) $y = 2\left[\frac{1}{4}(x + 1)\right] - 6$

[11, 15[

b) $y = -3[2(x - 4)] - 12$

[2, 2.5[

c) $y = 4[2x] + 2$

No zero

d) $y = -5[x - 8]$

[8, 9[

3. Determine the initial value of the function $f(x) = -3\left[\frac{1}{5}(x-9)\right] + 10$ ***16***

4. Determine over what interval the function $f(x) = 3\left[\frac{1}{4}(x-1)\right] + 6$ is positive. ***$[-7, +\infty[$***

5. Determine over what interval the function $f(x) = 5[x-3] + 1$ is strictly negative. ***$]-\infty, 3[$***

6. Determine over what interval the function $f(x) = 3\left[\frac{1}{2}(x-7)\right] + 6$ is increasing. ***$f \nearrow$ over $\mathbb{R}$***

7. Consider the functions $f(x) = 2\left[\frac{1}{4}(x-1)\right] + 2$ and $g(x) = -3\sqrt{x+5} + 4$.

Determine $g \circ f(7) =$ ***-5***

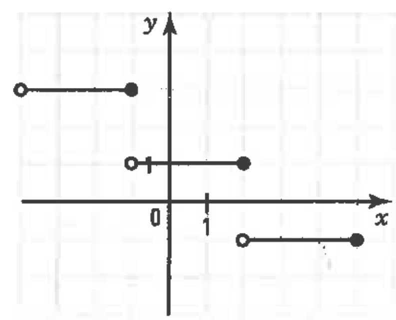
8. Determine the rule of the greatest integer function represented on the right.

We choose $(h, k) = (2, 1)$

○—● implies that $b < 0$, $b = -\frac{1}{3}$

$f \searrow$ implies that $a > 0$, $a = 2$

Rule: $y = 2\left[-\frac{1}{3}(x-2)\right] + 1$



9. A salesman in a store receives a weekly base salary of \$300 plus a commission of \$40 for every 10 items he sells during that week.

a) Find the rule of the function which gives the salesman's salary y as a function of the number of items sold x. ***$y = 40\left[\frac{x}{10}\right] + 300$***

b) What is this salesman's salary if he sold 84 items this week? ***\$620***

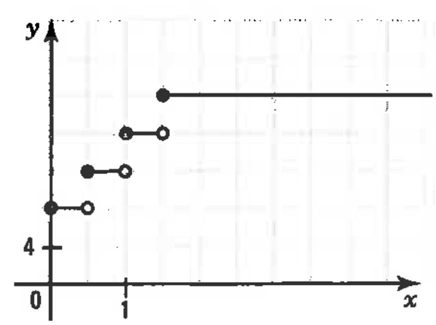
c) In what interval is the number of items sold if the salesman's salary is \$500?

In the interval $[50, 60[$

d) Can this salesman earn a salary of \$450? Justify your answer.

No, the equation $40\left[\frac{x}{10}\right] + 300 = 450$ has no solution since $\left[\frac{x}{10}\right] \neq 3.75$.

10. The cost of parking in a lot is \$8 for a duration of less than 30 min. Afterward, the cost increases by \$4 for every 30 minutes or part thereof. The maximum cost is \$20 per day.



a) What is the rule of the function which gives the cost y (in \$) as a function of the parking duration x (in hours).

$y = 4[2x] + 8$

b) Represent this situation in the Cartesian plane on the right.

c) What is the cost for a parking duration of 1 h 40 min? ***\$20***

d) In what interval is the parking duration if the cost is \$12?

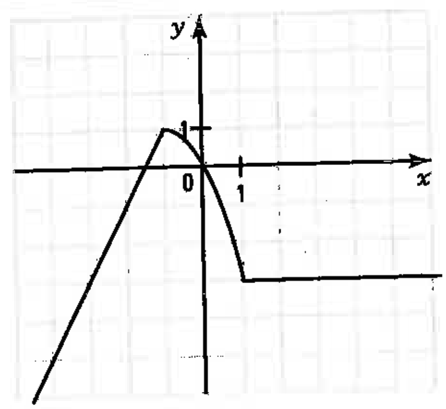
In the interval $\left[\frac{1}{2}, 1\right[$

3.6 Piecewise function

ACTIVITY 1 Graph of a piecewise function

A function f is defined by three different rules, depending on the interval over which x is located.

- Over the interval $]-\infty, -1]$, the function f is defined by the rule $f(x) = 2x + 3$.
- Over the interval $]-1, 1]$, the function f is defined by the rule $f(x) = -(x + 1)^2 + 1$.
- Over the interval $]1, +\infty]$, the function f is defined by the rule $f(x) = -3$.



a) Represent, in the Cartesian plane on the right, the function f.

b) Determine

1. $f(-2) =$ ***-1***
2. $f(0) =$ ***0***
3. $f(5) =$ ***-3***

c) Find

1. dom $f =$ $\mathbb{R}$
2. ran $f =$ ***]-∞, 1]***

d) Find

1. the zero of f. ***0***
2. the initial value of f. ***0***

e) Determine over what interval the function is positive. $\left[-\frac{3}{2}, 0\right]$

f) Determine over what interval the function is

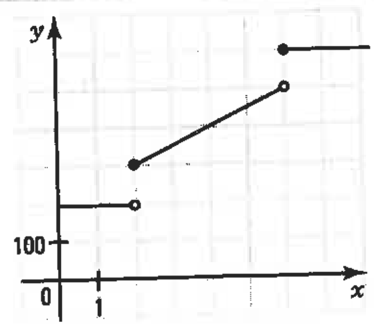
1. strictly increasing. ***]-∞, -1]***
2. strictly decreasing. ***[-1, 1]***
3. constant. ***[1, +∞[***

g) Does the function f have any extrema? If yes, what? ***Yes, a maximum; max f = 1***

ACTIVITY 2 An employee's salary

The weekly salary $f(x)$ of an employee in an electronic games store is calculated, according to the number x of games sold, using the following rule:

$$f(x) = \begin{cases} 200 & \text{if } x < 2 \\ 50x + 200 & \text{if } 2 \leqslant x < 6 \\ 600 & \text{if } x \geqslant 6 \end{cases}$$



a) What is the salary of an employee who sells

1. 2 games? ***300 $***
2. 4 games? ***400 $***
3. 12 games? ***600 $***

b) Determine the number of games sold by an employee whose salary is

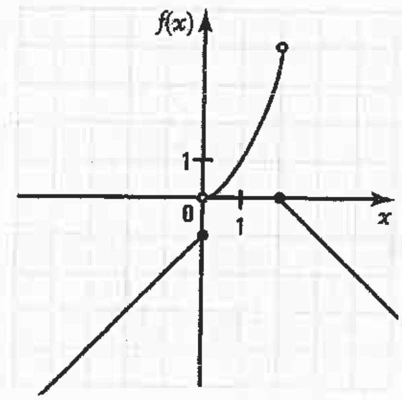
1. \$200. ***0 or 1 game sold***
2. \$450. ***5 games***
3. \$600. ***6 games or more***

PIECEWISE FUNCTIONS

A piecewise function is a function whose rule differs depending on the interval over which the variable x is located.

Ex.: Consider the following function.

$$f(x) = \begin{cases} x - 1 & \text{if } x \leqslant 0 \\ x^2 & \text{if } 0 < x < 2 \\ -x + 2 & \text{if } x \geqslant 2 \end{cases}$$



The graph of this function is represented in the Cartesian plane on the right.

$\text{dom } f = \mathbb{R}$, $\text{ran } f =]-\infty, 4[$

When we evaluate this function for a given value of the variable x, we find in which interval this value belongs to and we use the rule of the function defined over this interval.

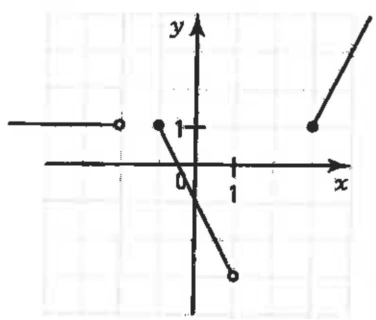
Thus, $f(1.5) = (1.5)^2 = 2.25$; $f(3) = -(3) + 2 = -1$.

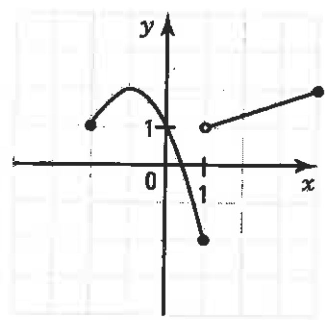
1. Graph the following functions.

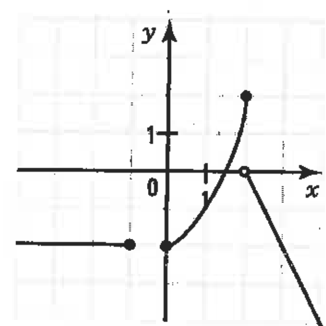
a) $f_1(x) = \begin{cases} 1 & \text{if } x < -2 \\ -2x - 1 & \text{if } -1 \leqslant x < 1 \\ 2x - 5 & \text{if } x \geqslant 3 \end{cases}$

b) $f_2(x) = \begin{cases} -(x+1)^2 + 2 & \text{if } -2 \leqslant x \leqslant 1 \\ \dfrac{1}{3}x + \dfrac{2}{3} & \text{if } 1 < x \leqslant 4 \end{cases}$

c) $f_3(x) = \begin{cases} -2 & \text{if } x \leqslant -1 \\ x^2 & \text{if } 0 \leqslant x \leqslant 2 \\ -2x + 4 & \text{if } x > 2 \end{cases}$







2. For each of the piecewise functions given in number 1, find

a) the domain and range.

dom f_1 = $]-\infty, -2[\cup [-1, 1[\cup [3, +\infty[$; ran f_1 = $]-3, +\infty[$

dom f_2 = $[-2, 4]$; ran f_2 = $[-2, 2]$

dom f_3 = $]-\infty, -1] \cup [0, +\infty[$; ran f_3 = $]-\infty, 2]$

b) the image of 2.

$f_1(2)$: does not exist; $f_2(2)$: $\frac{4}{3}$; $f_3(2)$: 2

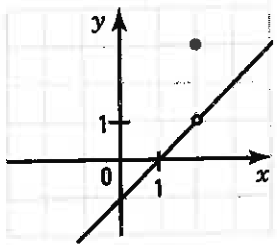
c) the initial value.

$y_1 = -1$; $y_2 = 1$; $y_3 = -2$

3. Graph the following functions and determine their domain.

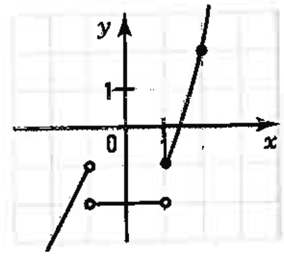
a) $f(x) = \begin{cases} x - 1 & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$

dom f = ℝ



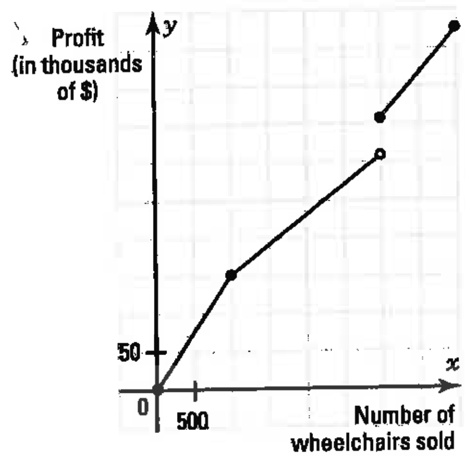
b) $f(x) = \begin{cases} 2x + 1 & \text{if } x < -1 \\ -2 & \text{if } -1 < x < 1 \\ x^1 - 2 & \text{if } x \geq 2 \end{cases}$

dom f = ℝ \{-1}



4. The Kandev company sells wheelchairs to residences for the elderly. The function f which gives the annual net profit y (in thousands of dollars) as a function of the number x of wheelchairs sold is given by the rule:

$$f(x) = \begin{cases} 0.15x & 0 \leq x \leq 1000 \\ 0.08x + 70 & 1000 < x < 3000 \\ 0.12x & 3000 \leq x \leq 4000 \end{cases}$$



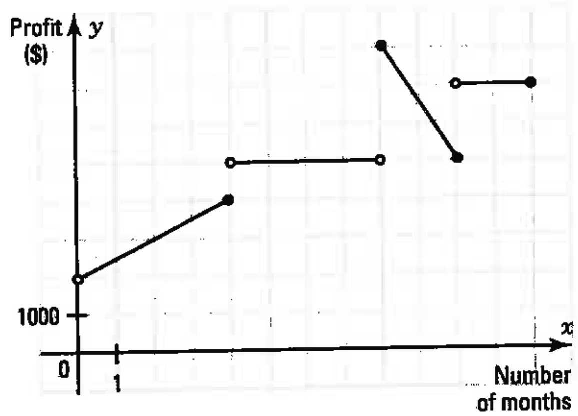
a) If the maximum number of wheelchairs sold per year is 4000, draw the graph of this function.

b) Find dom f. *[0, 4000]*

c) What is the profit made from selling 2500 wheelchairs? *$270*

d) Over what interval is the rate of change the greatest? *[0, 1000]*

5. The piecewise function f represented on the right gives a company's accumulated profit $f(x)$ as a function of the number x of elapsed months.



a) What is the company's accumulated profit after

1. 2 months? *$3000* 2. 4 months? *$4000*
3. 6 months? *$5000* 4. 11 months? *$7000*

b) Determine the number of elapsed months if the company's accumulated profit is

1. $3000. *2 months*
2. $6500. *9 months*

c) Determine the rule of the function f.

$$f(x) = \begin{cases} 500x + 2000 & \text{if } 0 < x \leq 4 \\ 5000 & \text{if } 4 < x < 8 \\ -1500x + 20\,000 & \text{if } 8 \leq x \leq 10 \\ 7000 & \text{if } 10 < x \leq 12 \end{cases}$$

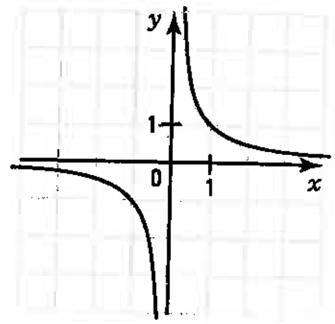
d) Over what interval is the function f

1. strictly increasing? *]0, 4]*
2. strictly decreasing? *[8, 10]*
3. constant? *]4,8[or]10,12]*

3.7 Rational function

Activity 1 Basic rational function

Consider the function f defined by the rule: $f(x) = \frac{1}{x}$.



a) What restriction must be imposed on the variable x?

x must be a non-zero real number.

b) Complete the following table.

x	-4	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$f(x)$	$-\frac{1}{4}$	$-\frac{1}{2}$	-1	-2	-4		4	2	1	$\frac{1}{2}$	$\frac{1}{4}$

c) Indicate what number the variable y approaches as

1. the variable x takes positive values that are bigger and bigger. *0*
2. the variable x takes negative values that are smaller and smaller. *0*

d) Indicate the behaviour of the variable y as

1. the variable x takes positive values closer and closer to zero.
   The variable y takes bigger and bigger positive values.
2. the variable x takes negative values closer and closer to zero.
   The variable y takes smaller and smaller negative values.

e) Graph the function in the Cartesian plane above.

f) Observe the branch of the hyperbola located in the 1st quadrant.

1. When x takes positive values that are bigger and bigger, the branch gets closer and closer to the x-axis without ever touching it. *We say that the x-axis is a **horizontal asymptote** to the curve.* What is the equation of this asymptote? *$y = 0$*
2. When x takes positive values closer and closer to zero, the branch gets closer and closer to the y-axis without ever touching it. *We say that the y-axis is a **vertical asymptote** to the curve.* What is the equation of this asymptote? *$x = 0$*

g) Observe the branch of the hyperbola located in the 3rd quadrant.

1. Do we observe a horizontal asymptote? If yes, what is its equation?
   Yes; $y = 0$
2. Do we observe a vertical asymptote? If yes, what is its equation?
   Yes; $x = 0$

The represented curve is called a **hyperbola**. This hyperbola consists of two branches. Place a random point $M(x, y)$ on a branch and verify that the point $M'(-x, -y)$ is located on the other branch. The origin O, mid-point of the segment MM′, is therefore called the **symmetric centre** of the hyperbola.

h) Determine

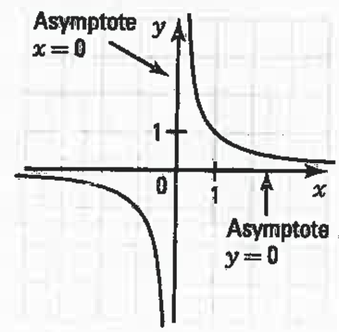
1. dom $f =$ $\mathbb{R}^*$
2. ran $f =$ $\mathbb{R}^*$
3. the zero of f. ***does not exist***
4. the initial value of f. ***does not exist***
5. the sign of f. ***$f(x) \geq 0$ over $\mathbb{R}^*_+$; $f(x) < 0$ over $\mathbb{R}^*_-$.***
6. the variation of f. ***$f \searrow$ over $\mathbb{R}^*$; f is never increasing.***
7. the extrema of f (if it exists). ***does not exist***

BASIC RATIONAL FUNCTION

- Consider the rational function defined by the rule:

$$\boxed{f(x) = \frac{1}{x}}$$

This function is called the **basic rational function**.



- We have:
  - dom $f = \mathbb{R}^*$, ran $f = \mathbb{R}^*$.
  - f has no zeros.
  - f is decreasing over $\mathbb{R}^*$.
  - The represented curve is called a **hyperbola.** This hyperbola consists of two branches.
  - The origin 0 is the symmetrical centre of the hyperbola.
  - The hyperbola has two asymptotes: the x-axis and the y-axis.

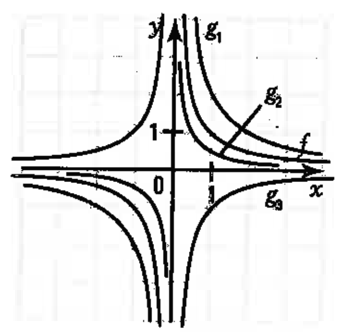
ACTIVITY 2 Role of the parameters a, b, h and k

The basic rational function $f(x) = \frac{1}{x}$ can be transformed into a rational function with the rule

$$\boxed{g(x) = \frac{a}{b(x-h)} + k} \text{ (standard form)}$$

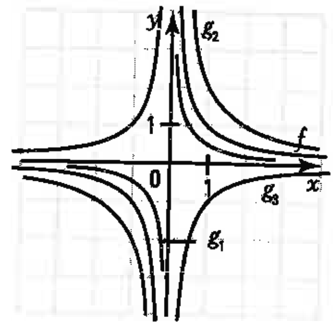
a) Consider the basic rational function $f(x) = \frac{1}{x}$ and the rational function $g(x) = \frac{a}{x}$.

Represent, in the same Cartesian plane, the functions $g_1(x) = \frac{2}{x}$, $g_2(x) = \frac{0.5}{x}$ and $g_3(x) = \frac{-1}{x}$ and explain how to deduce the graph of g from the graph of f when



1. $a > 1$: ***by a vertical stretch.***
2. $0 < a < 1$: ***by a vertical reduction.***
3. $a = -1$: ***by a reflection about the x-axis.***
4. Complete: From the graph of $f(x) = \frac{1}{x}$, we obtain the graph $g(x) = \frac{a}{x}$ by the transformation $(x, y) \rightarrow$ ***(x, ay)***.

b) Consider the basic rational function $f(x) = \frac{1}{x}$ and the rational function $g(x) = \frac{1}{bx}$.



Represent, in the same Cartesian plane, the functions $g_1(x) = \frac{1}{2x}$, $g_2(x) = \frac{1}{0.5x}$ and $g_3(x) = \frac{1}{-x}$ and explain how to deduce the graph of g from the graph of f when

1. $b > 1$: ***by a horizontal reduction.***
2. $0 < b < 1$: ***by a horizontal stretch.***
3. $b = -1$: ***by a reflection about the y-axis.***
4. Complete: From the graph of $f(x) = \frac{1}{x}$, we obtain the graph $g(x) = \frac{1}{bx}$ by the transformation $(x, y) \rightarrow$ $\left(\frac{x}{b}, y\right)$.
5. Compare the graphs of f and g in each of the following cases and justify your answer.

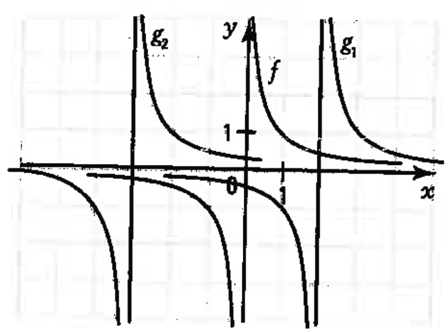
$f(x) = \frac{2}{x}$ and $g(x) = \frac{1}{0.5x}$: ***They are the same. In fact,*** $\frac{1}{0.5x} = \frac{1}{\frac{1}{2}x} = \frac{2}{x}$.

$f(x) = \frac{0.5}{x}$ and $g(x) = \frac{1}{2x}$: ***They are the same. In fact,*** $\frac{0.5}{x} = \frac{\frac{1}{2}}{x} = \frac{1}{2x}$.

$f(x) = \frac{-1}{x}$ and $g(x) = \frac{1}{-x}$: ***They are the same. In fact,*** $\frac{-1}{x} = \frac{1}{-x}$.

The reflection about the x-axis and the reflection about the y-axis have the same effect on the basic rational function.

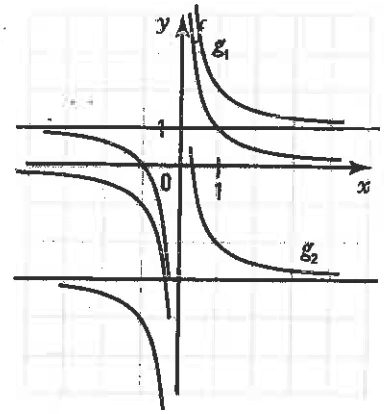
c) Consider the basic rational function $f(x) = \frac{1}{x}$ and the rational function $g(x) = \frac{1}{x - h}$.



Represent, in the same Cartesian plane, the functions $g_1(x) = \frac{1}{x - 2}$, $g_2(x) = \frac{1}{x + 3}$ and explain how to deduce the graph of g from the graph of f when

1. $h > 0$: ***by a horizontal translation to the right.***
2. $h < 0$: ***by a horizontal translation to the left.***
3. What is the equation of the vertical asymptote of the function $g(x) = \frac{1}{x - h}$? $x = h$
4. Complete: From the graph of $f(x) = \frac{1}{x}$, we obtain the graph $g(x) = \frac{1}{x - h}$ by the transformation $(x, y) \rightarrow$ $(x + h, y)$.

d) Consider the basic rational function $f(x) = \frac{1}{x}$ and the rational function $g(x) = \frac{1}{x} + k$.



Represent, in the same Cartesian plane, the functions $g_1(x) = \frac{1}{x} + 1$, $g_2(x) = \frac{1}{x} - 3$ and explain how to deduce the graph of g from the graph of f when

1. $k > 0$: ***by a vertical translation upward.***
2. $k < 0$: ***by a vertical translation downward.***
3. What is the equation of the horizontal asymptote of the function $g(x) = \frac{1}{x} + k$? ***$y = k$.***
4. Complete: From the graph of $f(x) = \frac{1}{x}$, we obtain the graph $g(x) = \frac{1}{x} + k$ by the transformation $(x, y) \rightarrow$ ***$(x, y + k)$***.

RATIONAL FUNCTION – STANDARD FORM

- The graph of the function

$$\boxed{f(x) = \frac{a}{b(x - h)} + k}$$

is deduced from the graph of the basic rational function $y = \frac{1}{x}$ by the transformation

$$\boxed{(x, y) \rightarrow \left(\frac{x}{b} + h,\ ay + k\right)}$$

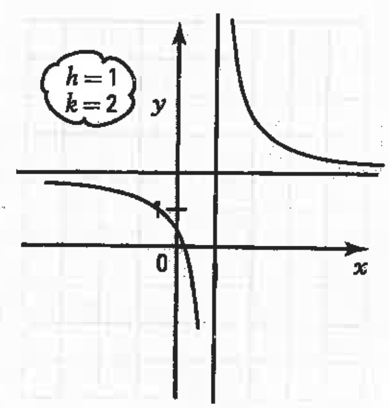
- This hyperbola has two asymptotes, the **vertical** asymptote with equation $x = h$ and the **horizontal** asymptote with equation $y = k$.
- The point (h, k) is the **symmetrical centre** of the hyperbola.

Ex.: To graph the hyperbola $y = \frac{3}{2(x - 1)} + 2$,

1. we draw the asymptotes.
   - vertical asymptote: $x = 1$.
   - horizontal asymptote: $y = 2$.
2. we complete a table of values.

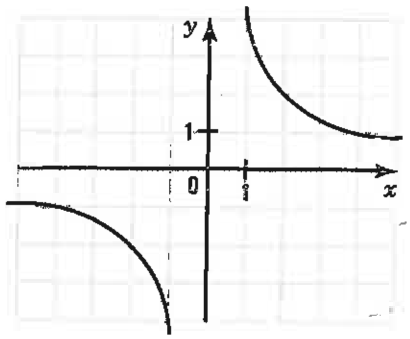
x	−2	−1	0	1	2	3	4
y	1.5	1.25	0.5		3.5	2.75	2.5

3. we draw the hyperbola using the symmetrical centre (h, k).

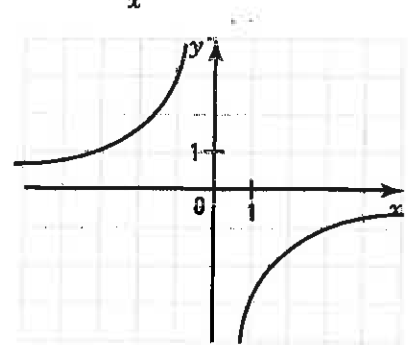


1. Graph the following rational functions.

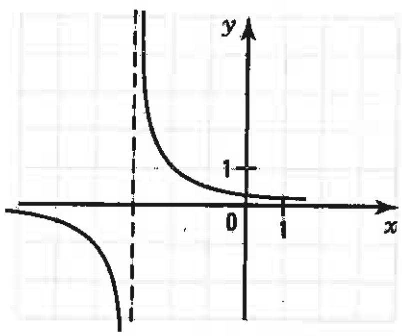
a) $f(x) = \frac{4}{x}$.



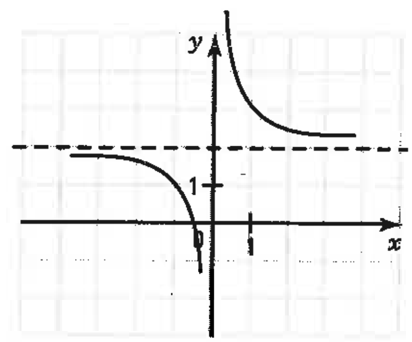
b) $f(x) = -\frac{3}{x}$.



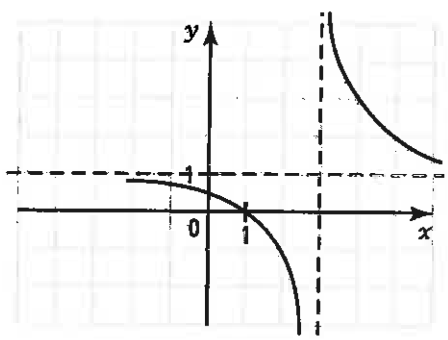
c) $f(x) = \frac{1}{x+3}$.



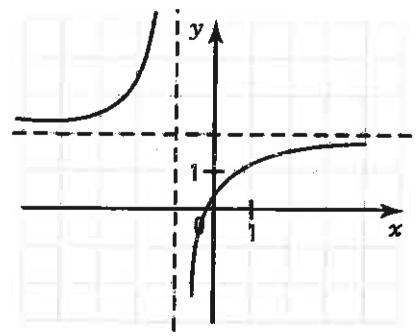
d) $f(x) = \frac{1}{x} + 2$.



e) $f(x) = \frac{2}{x-3} + 1$.

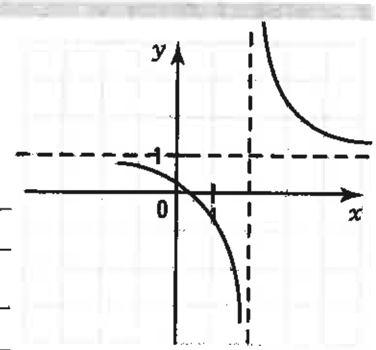


f) $f(x) = \frac{3}{-2(x+1)} + 2$.



ACTIVITY 3 Study of a rational function

Consider the function f defined by the rule $y = \frac{3}{2(x-2)} + 1$.



a) Graph the function in the Cartesian plane on the right.

b) Determine

1. dom f = *$\mathbb{R}\setminus\{2\}$* 2. ran f = *$\mathbb{R}\setminus\{1\}$*
3. the zero of f (if it exists). *0.5* 4. the initial value of f. *0.25*
5. the sign of f. *$f(x) \geq 0$ over $\left]-\infty, \frac{1}{2}\right] \cup \left]2, +\infty\right[$; $f(x) \leq 0$ over $\left[\frac{1}{2}, 2\right[$*
6. the variation of f. *$f \searrow$ over $\mathbb{R}\setminus\{2\}$; $f \nearrow$ never*
7. the extrema of f. *does not exist*

ACTIVITY 4 Finding the zero of a rational function

a) Consider the function defined by the rule: $y = \frac{-3}{4(x-2)} + 5$.
Justify the steps which enable you to find the zero of this function.

$\frac{-3}{4(x-2)} + 5 = 0$	*Replace y by zero.*
$\frac{-3}{4(x-2)} = -5$	*Subtract 5 from each side.*
$-20(x-2) = -3$	*The cross products are equal.*
$x - 2 = \frac{3}{20}$	*Divide each side by -20.*
$x = \frac{43}{20}$	*Add 2 to each side.*

b) Under what condition does the zero of a rational function defined by the rule $y = \frac{a}{b(x-h)} + k$ exist? *If $k \neq 0$*

STUDY OF A RATIONAL FUNCTION

Consider the rational function f defined by the rule:

$$\boxed{f(x) = \frac{a}{b(x-h)} + k} \text{ (standard form)}$$

- dom $f = \mathbb{R} \backslash \{h\}$; ran $f = \mathbb{R} \backslash \{k\}$
- The zero of f exists if $k \neq 0$, and the initial value of f exists if $h \neq 0$.
- To study the sign of f,
  - we find the zero (if it exists);
  - we establish the sign of f using a sketch of the graph.
- **Variation**
  - If $ab > 0$, f is decreasing over the domain.
  - If $ab < 0$, f is increasing over the domain.
- The rational function has no extrema.

2. Determine the domain and range of the following functions.

a) $y = \frac{-2}{4(x+5)} - 7$

dom = $\mathbb{R} \backslash \{-5\}$; ran = $\mathbb{R} \backslash \{-7\}$

b) $y = \frac{3}{2(x-1)} + 4$

dom = $\mathbb{R} \backslash \{1\}$; ran = $\mathbb{R} \backslash \{4\}$

3. Determine the zero and initial value of the following functions.

a) $y = \frac{3}{x-5} + 4$

Zero: $\frac{17}{4}$; i.v.: $\frac{17}{5}$

b) $y = \frac{-2}{3(x+1)}$

Zero: none; i.v.: $-\frac{2}{3}$

c) $y = \frac{-5}{4x} + 10$

Zero: $\frac{1}{8}$; i.v.: none

4. Determine the interval over which the function $f(x) = \frac{-4}{5(x-1)} + 3$ is positive. $[-\infty, 1[\cup \left[\frac{19}{15}, +\infty\right[$

5. Determine the interval over which the function $f(x) = \frac{3}{2(x+2)} - 1$ is strictly positive. $]-2, \frac{-1}{2}[$

6. Study the variation of the function $f(x) = \frac{-2}{5(x-1)} + 4$. ***f ↗ over $\mathbb{R} \setminus \{1\}$***

7. Consider the functions $f(x) = -2|-x + 4| + 5$, $g(x) = 3\sqrt{x-3} + 2$, $h(x) = \frac{6}{5(x-1)} + \frac{22}{5}$ and $i(x) = 3\left[\frac{1}{4}(x-2)\right] + 1$. Determine $f \circ g \circ h \circ i\,(1) =$ ***3***

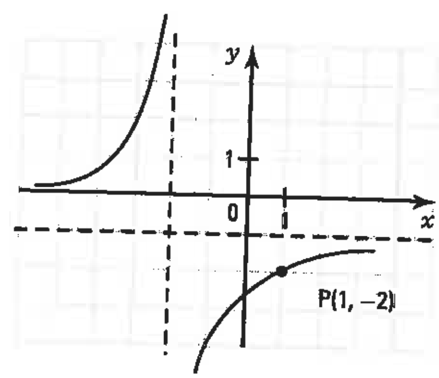
8. Given $f(x) = \frac{3}{2(x-4)} + 1$ and $g(x) = 3x - 1$. Determine, in standard form, the rule of the function $f \circ g$. ***$f \circ g(x) = f(3x-2) = \frac{3}{2(3x-6)} + 1 = \frac{1}{2(x-2)} + 1$.***

ACTIVITY 5 Finding the rule of a rational function

Any rule of a rational function can be written in the form $y = \frac{a}{x-h} + k$.

a) Consider the function $y = \frac{-3}{6(x-2)} + 1$. Write the rule of this function in the form $y = \frac{a}{x-h} + k$. ***$y = \frac{-0.5}{x-2} + 1$***

b) Consider a rational function whose graph passes through the point P(1, –2).

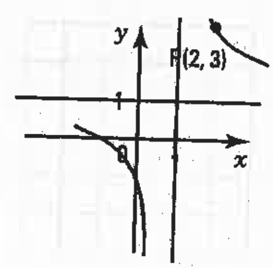


1. Identify h and k. ***$h = -2$, $k = -1$***
2. Determine a knowing that the coordinates of the point P(1, –2) verify the rule of the function.
   We have: ***$y = \frac{a}{x+2} - 1$; $-2 = \frac{a}{1+2} - 1$; $-1 = \frac{a}{3}$; $a = -3$.***
3. What is the rule of the function? ***$y = \frac{-3}{x+2} - 1$***

FINDING THE RULE OF A RATIONAL FUNCTION

Any rule of a rational function can be written in the form

$$\boxed{y = \frac{a}{x-h} + k}$$

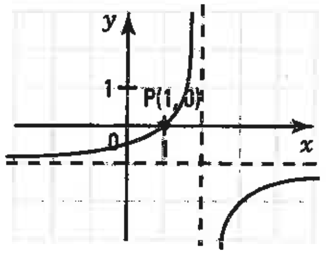


The asymptotes and a point are known.

1. Identify the parameters h and k. — 1. $h = 1$ and $k = 1$; $y = \frac{a}{x-1} + 1$
2. Find a after replacing, in the rule, x and y by the coordinates of the given point P. — 2. $3 = \frac{a}{2-1} + 1$; $a = 2$
3. Deduce the rule. — 3. $y = \frac{2}{x-1} + 1$

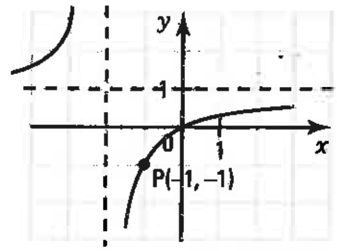
9. Find the rule of the following rational functions.

a)



$y = \frac{-1}{x-2} - 1$

b)



$y = \frac{-2}{x+2} + 1$

ACTIVITY 6 Inverse of a rational function – Standard form

a) Consider the rational function f defined by the rule: $y = \frac{-2}{3(x+1)} - 5$.

Justify the steps which enable you to determine the rule of the inverse function f^{-1}.

1. Isolate x in the equation $y = \frac{-2}{3(x+1)} - 5$

$y + 5 = \frac{-2}{3(x+1)}$ — ***Add 5 to each side.***

$3(x+1) = \frac{-2}{y+5}$ — ***Switch the extremes.***

$x + 1 = \frac{-2}{3(y+5)}$ — ***Divide each side by 3.***

$x = \frac{-2}{3(y+5)} - 1$ — ***Subtract 1 from each side.***

2. Interchange the letters x and y to obtain the rule of the inverse. We get:

$y = \frac{-2}{3(x+5)} - 1$

b) Complete: The inverse of a rational function is a ***rational function.***

c) 1. Determine

1) dom f = $\mathbb{R} \setminus \{-1\}$ 2) ran f = $\mathbb{R} \setminus \{-5\}$

3) dom f^{-1} = $\mathbb{R} \setminus \{-5\}$ 4) ran f^{-1} = $\mathbb{R} \setminus \{-1\}$

2. Verify that dom f^{-1} = ran f and that ran f^{-1} = dom f.

INVERSE OF A RATIONAL FUNCTION

The inverse of a rational function is a rational function.

Ex.: Given the rational function defined by the rule $y = \frac{-2}{3(x+1)} - 5$.

The inverse f^{-1} is a rational function defined by the rule $y = \frac{-2}{3(x+5)} - 1$.

(See activity 6 for finding the rule of f^{-1})

Note that dom f = ran $f^{-1} = \mathbb{R}\backslash\{1\}$ and that ran f = dom $f^{-1} = \mathbb{R}\backslash\{5\}$

10. Determine the inverse of the following rational functions.

a) $y = \frac{3}{x+5} - 1$ $y = \frac{3}{x+1} - 5$

b) $y = \frac{-1}{2(x-4)} + 3$ $y = \frac{-1}{2(x-3)} + 4$

11. A train travels a distance of 240 km. We consider the function f which gives the duration t (in h) of the trip as a function of the train's speed v (in km/h).

v (km/h)	40	60	80	120	160
t (h)	6	4	3	2	1.5

a) Complete the table of values on the right.

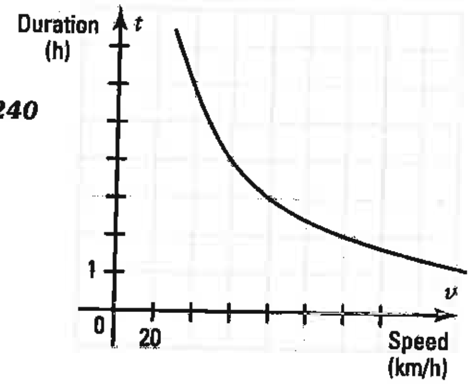
b) Is the rate of change of the function f constant? *No*

c) Verify that the product of the variables vt is constant. $vt = 240$

We say that the duration of the trip is **inversely proportional** to the speed or that the speed is inversely proportional to the duration.

d) What is the rule of the function? $t = \frac{240}{v}$

e) Graph the function f in the Cartesian plane.



f) Determine

1. dom f. $]0, +\infty[$
2. ran f. $]0, +\infty[$

g) When one variable increases, does the other variable increase or decrease? *It decreases.*

h) Is the function f increasing or decreasing? Justify your answer.
Decreasing, since the duration decreases as the speed increases.

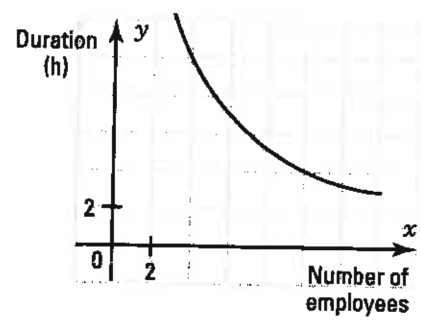
12. Renovations to a home require a total of 40 h of work for one employee. Consider the function f which gives the duration y (in h) of work per employee as a function of the number of employees x hired to do the renovations.

a) Complete the following table of values.

x	1	2	4	5	8	10
y	40	20	10	8	5	4

b) What is the rule of function f? $y = \frac{40}{x}$

c) Graph the function f in the Cartesian plane.



d) Is the function f increasing or decreasing? *Decreasing*

Activity 7 Rational function – General form

Consider the rational function defined by the rule $y = \frac{3}{2(x-5)} + 4$ (standard form).

a) Justify the steps which enable you to write the rule of this function in the form $y = \frac{ax+b}{cx+d}$.

$y = \frac{3}{2(x-5)} + 4 = \frac{3}{2(x-5)} + \frac{8(x-5)}{2(x-5)}$ ***Finding a common denominator***

$= \frac{3+8(x-5)}{2(x-5)}$ ***Addition of the 2 fractions; $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$.***

$= \frac{8x-37}{2x-10}$ ***Simplification***

*The form $y = \frac{ax+b}{cx+d}$ is called the **general form** of a rational function.*

b) 1. Identify the parameters h and k of the standard form. ***$h = 5$; $k = 4$***

2. Identify the parameters a, b, c and d of the general form. ***$a = 8$, $b = -37$, $c = 2$, $d = -10$***

3. Verify that the vertical asymptote has the equation $x = -\frac{d}{c}$. ***$x = h = 2$ and $x = -\frac{d}{c} = 2$***

4. Verify that the horizontal asymptote has the equation $y = \frac{a}{c}$. ***$y = k = 4$ and $y = \frac{a}{c} = 4$***

c) Consider the rational function $y = \frac{5x-3}{2x+4}$ (general form).

To obtain the standard form from the general form $y = \frac{A(x)}{B(x)}$ where $A(x) = 5x - 3$ and $B(x) = 2x + 4$, we proceed in the following manner:

1° Determine the quotient $Q(x)$ and the remainder $R(x)$ from Euclidean division (i.e. long division) of $A(x)$ by $B(x)$.

$$\begin{array}{r|l} A(x) & B(x) \\ \hline R(x) & Q(x) \end{array}$$

2° From the Euclidean relation $A(x) = B(x) \cdot Q(x) + R(x)$, we deduce the standard form of the rule.

$$\frac{A(x)}{B(x)} = Q(x) + \frac{R(x)}{Q(x)}$$

1. Perform the Euclidean division of $A(x) = 5x - 3$ by $B(x) = 2x + 4$ and determine the quotient $Q(x)$ and the remainder $R(x)$.

$$\begin{array}{r|l} 5x - 3 & 2x + 4 \\ \hline -\;5x + 10 & \frac{5}{2} \\ \hline -13 & \end{array}$$

$Q(x) = \frac{5}{2}$; $R(x) = -13$

2. Deduce the standard form of the rule of the function $y = \frac{5x-3}{2x+4}$. ***$y = \frac{5}{2} + \frac{-13}{2(x+2)}$***

RATIONAL FUNCTION – GENERAL FORM

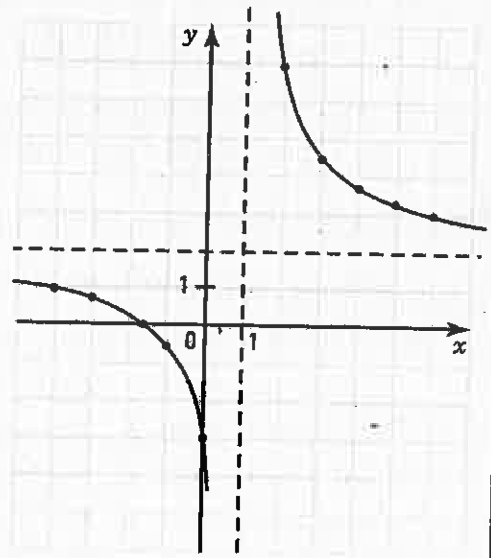
- The general form of a rational function is:

$$\boxed{f(x)=\frac{ax+b}{cx+d}}$$

- $\operatorname{dom} f=\mathbb{R}\setminus\left\{-\frac{d}{c}\right\}$; $\operatorname{ran} f=\mathbb{R}\setminus\left\{\frac{a}{c}\right\}$.
- Vertical asymptote: $x=-\frac{d}{c}$; horizontal asymptote: $y=\frac{a}{c}$.

Ex.: Given the rational function $f(x)=\frac{2x+3}{x-1}$.

- $\operatorname{dom} f=\mathbb{R}\setminus\{1\}$; $\operatorname{ran} f=\mathbb{R}\setminus\{2\}$.
- Vertical asymptote: $x=1$;
  horizontal asymptote: $y=2$.
- Zero of f: $f(x)=0\Leftrightarrow 2x+3=0\Leftrightarrow x=-\frac{3}{2}$.
- Sign of f: $f(x)\geqslant 0\Leftrightarrow x\in\left]-\infty,-\frac{3}{2}\right]\cup\left]1,+\infty\right[$
  $f(x)\leqslant 0\Leftrightarrow x\in\left[-\frac{3}{2},1\right[$.
- Variation of f: f is decreasing over $\mathbb{R}\setminus\{1\}$.
- f has no extrema.



13. Determine the domain and range of the following rational functions.

a) $y=\frac{3x+2}{x-5}$

***dom** $=\mathbb{R}\setminus\{5\}$, **ran** $f=\mathbb{R}\setminus\{3\}$*

b) $y=\frac{-2x+4}{3x-6}$

***dom** $=\mathbb{R}\setminus\{2\}$, **ran** $f=\mathbb{R}\setminus\left\{-\frac{2}{3}\right\}$*

c) $y=\frac{5x+4}{2x-3}$

***dom** $=\mathbb{R}\setminus\left\{\frac{3}{2}\right\}$, **ran** $f=\mathbb{R}\setminus\left\{\frac{5}{2}\right\}$*

14. Determine the zero (if it exists) and the initial value (if it exists) of the following functions.

a) $y=\frac{3x-2}{x-4}$

***Zero:** $\frac{2}{3}$, **i.v:** 3*

b) $y=\frac{-5x+10}{2x-5}$

***Zero:** 2, **i.v:** -2*

c) $y=\frac{-2x-6}{4x}$

***Zero:** -3, **i.v:** does not exist*

15. Determine over which interval the following functions are positive.

a) $y=\frac{4x+2}{x-3}$

***$f(x)\geqslant 0$ over** $\left]-\infty,\frac{-1}{2}\right]\cup\left]3,+\infty\right[$*

b) $y=\frac{-2x+8}{4x-2}$

***$f(x)\geqslant 0$ over** $\left]\frac{1}{2},4\right]$*

16. Study the variation of the following functions.

a) $y=\frac{-4x+9}{x-3}$

*$f\nearrow$ **over** $\mathbb{R}\setminus\{3\}$*

b) $y=\frac{2x+5}{3x-2}$

*$f\searrow$ **over** $\mathbb{R}\setminus\left\{\frac{2}{3}\right\}$*

17. Write the rule of the following rational functions in general form.

a) $y=\frac{3}{2(x-1)}+4$ $y=\frac{8x-5}{2x-2}$

b) $y=\frac{-2}{5(x-3)}-1$ $y=\frac{-5x+13}{5x-15}$

18. Write the rule of the following rational functions in standard form.

a) $y=\frac{3x+2}{x-3}$ $y=\frac{11}{x-3}+3$

b) $y=\frac{4x+3}{2x-6}$ $y=\frac{15}{2(x-3)}+2$

c) $y=\frac{-2x+5}{3x+4}$ $y=\frac{23}{9\left(x+\frac{4}{3}\right)}-\frac{2}{3}$

19. Consider the rational functions $f(x)=\frac{2x+3}{x-4}$ and $g(x)=\frac{3x+5}{x+3}$.

a) Determine the rule of the composite

1. $g\circ f(x)=$ $y=\frac{11x-11}{5x-9}$
2. $f\circ g(x)=$ $y=\frac{9x+19}{-x-7}$

b) What can you say about the composition of a rational function with a rational function?

The composition of a rational function with a rational function is also a rational function.

20. Consider the rational function $y=\frac{5x+4}{x-3}$ (general form).

Justify the steps which enable you to determine the rule of the inverse f^{-1}.

1. Isolate x in the equation $y=\frac{5x+4}{x-3}$.

$y(x-3)=5x+4$ *Cross products are equal.*

$xy-3y=5x+4$ *Distributive property of multiplication over subtraction.*

$xy-5x=3y+4$ *Subtract 5x and add 3y to each side.*

$x(y-5)=3y+4$ *Factor out x on the left side.*

$x=\frac{3y+4}{y-5}$ *Isolate the variable x.*

2. Switch the letters x and y to obtain the rule of the inverse.

We get: $y=\frac{3x+4}{x-5}$.

21. Consider the rational function $f(x)=\frac{3x-2}{2x+5}$.

a) Determine the rule of the inverse f^{-1}. $f^{-1}(x)=\frac{-5x-2}{2x-3}$

b) Verify that

1. $f\circ f^{-1}(x)=x$
2. $f^{-1}\circ f(x)=x$

22. Consider the rational function $f(x)=\frac{-2x+3}{4x+1}$.

a) Determine the domain and range of f. $\text{dom } f=\mathbb{R}\setminus\left\{-\frac{1}{4}\right\}$, $\text{ran } f=\mathbb{R}\setminus\left\{-\frac{1}{2}\right\}$

b) Determine the rule of the inverse f^{-1}. $f^{-1}(x)=\frac{-x+3}{4x+2}$

c) Determine the domain and range of the inverse f^{-1} and verify that dom f^{-1} = ran f and ran f^{-1} = dom f.

$\text{dom } f^{-1}=\mathbb{R}\setminus\left\{-\frac{1}{2}\right\}$, $\text{ran } f^{-1}=\mathbb{R}\setminus\left\{-\frac{1}{4}\right\}$

Evaluation 3

1. Determine the domain and range of each of the following functions.

a) $y = -2x^2 + 4x - 9$

dom = ℝ, ran =]−∞, −7]

b) $y = 4|x - 5| + 8$

dom = ℝ, ran = [8, +∞[

c) $y = \frac{1}{2}\sqrt{-(x-4)} + 3$

dom =]−∞, 4], ran = [3, +∞[

d) $y = -2\left[\frac{1}{3}(x-5)\right] + 4$

dom = ℝ,
ran = {y | y = −2m + 4, m ∈ ℤ}

e) $y = \frac{4}{3(x-1)} + 2$

dom = ℝ \ {1}, ran = ℝ \ {2}

f) $y = -4x + 2$

dom = ℝ, ran = ℝ

2. Determine the zero(s) and the initial value of each of the following functions.

a) $y = -2(x-4)^2 + 8$

Zeros: 2 and 6, i.v.: −24

b) $y = 3x - 5$

Zero: $\frac{5}{3}$, i.v.: −5

c) $y = \frac{3}{4}\sqrt{x+1} - 3$

Zero: 15, i.v.: $\frac{-9}{4}$

d) $y = 3\left[\frac{1}{2}(x-5)\right] + 6$

Zeros: [1, 3[, i.v.: −3

e) $y = \frac{-2}{5(x-1)} + 4$

Zero: $\frac{11}{10}$, i.v.: $\frac{22}{5}$

f) $y = 3|2x - 1| - 6$

Zeros: $\frac{-1}{2}$ and $\frac{3}{2}$, i.v.: −3

3. Determine over what interval each of the following functions is negative.

a) $y = 2x^2 - 5x - 3$

$\left]-\frac{1}{2}, 3\right[$

b) $y = -7x + 63$

[9, +∞[

c) $y = 2|8 - x| - 12$

[2, 14]

d) $y = \frac{2}{x-5} + 4$

$\left]\frac{9}{2}, 5\right[$

e) $y = -2\sqrt{6-x} + 4$

]−∞, 2]

f) $y = -\left[\frac{x}{2}\right] - 3$

[−6, +∞[

4. Determine over what interval each of the following functions is increasing.

a) $y = -3(x-5)(x+1)$

]−∞, 2]

b) $y = 2x - 5$

ℝ

c) $y = -[6 - 3x] + 1$

ℝ

d) $y = -3\sqrt{-(x-1)} + 4$

]−∞, 1]

e) $y = 3|x - 5| + 2$

[5, +∞[

f) $y = \frac{3}{2(x-1)} + 5$

∅

5. Determine, if it exists, the extremum of each of the following functions.

a) $y = -3x^2 + 12x - 7$

max = 5

b) $y = -2|3 - 2x| + 5$

max = 5

c) $y = -2\sqrt{x} + 7$

max = 7

6. Find the rule of the inverse of each of the following functions.

a) $y = -3x + 8$

$y = -\frac{1}{3}x + \frac{8}{3}$

b) $y = 3\sqrt{2-x} + 4$

$y = -\frac{1}{9}(x-4)^2 + 2,\ x \geq 4$

c) $y = \frac{3}{2(x-1)} + 8$

$y = \frac{3}{2(x-8)} + 1$

7. Consider the following real functions.

$f(x) = 3x - 8$ $\quad g(x) = 3\sqrt{2x+1} - 5$ $\quad h(x) = -2|x - 4| + 12$

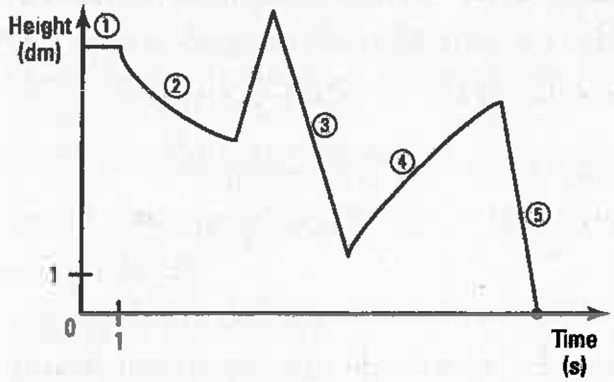
$i(x) = 3(x - 2)^2 + 4$ $\quad k(x) = \dfrac{2}{x-5} + 1$ $\quad l(x) = 3\left[\frac{1}{5}(x+4)\right] - 6$

Determine

a) $f \circ g(4) =$ ***4*** **b)** $l \circ h(3) =$ ***0*** **c)** $k \circ i(5) =$ $\boldsymbol{\frac{14}{13}}$

d) $f \circ l(0) =$ ***-26*** **e)** $k \circ f \circ h(2) =$ $\boldsymbol{\frac{13}{11}}$ **f)** $l \circ h(-6) =$ ***-9***

8. The path of a marble in a child's game can be represented by the graph in the Cartesian plane below. Initially, the marble is at a height of 7 dm from the ground.



$$f(x) = \begin{cases} 7 & \text{if } 0 \leq x < 1 \\ \frac{3}{x} + 4 & \text{if } 1 \leq x \leq 4 \\ a|x-5| + 8 & \text{if } 4 \leq x \leq 7 \\ 2\sqrt{x-7} + k & \text{if } 7 \leq x \leq 11 \\ -5.5x + b & \text{if } 11 \leq x \leq t \end{cases}$$

Determine the duration t of the marble's path.

f(4) = 4.75: $\quad$ ***a |4 − 5| + 8 = 4.75;*** $\quad$ ***a = −3.25***

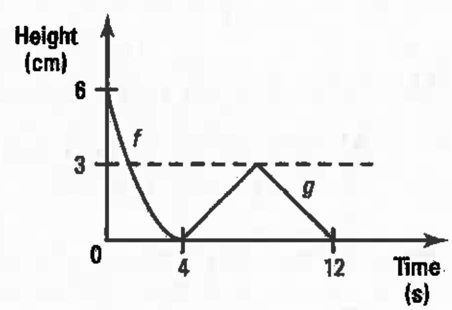
f(7) = 1.5; k = 1.5; $\quad$ ***f(11) =*** $\boldsymbol{2\sqrt{11-7} + 1.5 = 5.5}$

−5.5(11) + b = 5.5; b = 66; −5.5 t + 66 = 0 ⇒ t = 12 s.

9. Aaron is playing an electronic game. The height of a flashing dot on the screen can be modeled by a square root function f from 0 to 4 seconds and by an absolute value function g from 4 to 12 seconds as indicated by the graph on the right.

The starting point of the flashing dot is the vertex of the function f.

Determine at what times the flashing dot is at a height of 1.5 cm.



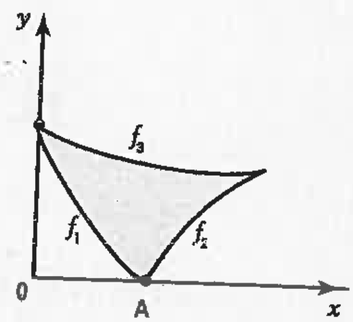
$\boldsymbol{f(x) = -3\sqrt{x} + 6\,;\; g(x) = -\frac{3}{4}|x - 8| + 3}$

The flashing dot is at a height of 1.5 cm at the times t = 2.25 s, t = 6 s and t = 10 s.

10. A company's logo was drawn using the graphs of three square root functions as indicated in the figure on the right.

The rules of the functions f_1 and f_3 are respectively $f_1(x) = -\frac{4}{3}\sqrt{x} + 4$ and $f_3(x) = -\frac{1}{4}\sqrt{x} + 4$.

The x-coordinate of the intersection point of the functions f_2 and f_3 is 16. Knowing that point A is the vertex of the function f_2, what is the rule of the function f_2?



$f_3(16) = 3$; $A(9, 0)$; f_2: $y = a\sqrt{x - 9}$

The rule of the function f_2 is: $y = \frac{3}{7}\sqrt{7(x - 9)}$

11. The value of one Kandev share fluctuated, over a one-month period, according to the rule of an absolute value function. At the opening of the market, this share was worth \$3.50. Twelve days later, it reaches its maximum value of \$8.

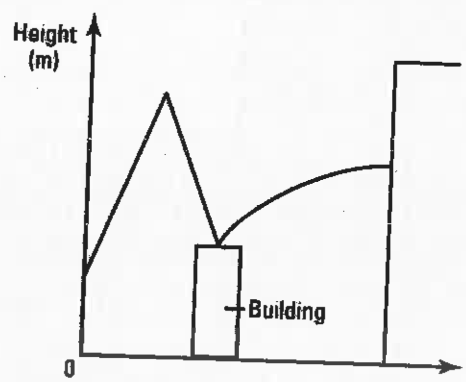
How many days go by between the moment the value of the share is worth \$5 for the first time and the moment it is worth \$2 on its descent?

$y = -\frac{3}{8}|x - 12| + 8$; $-\frac{3}{8}|x - 12| + 8 = 5$; $-\frac{3}{8}|x - 12| + 8 = 2$.

24 days.

12. The graph on the right illustrates a projectile's trajectory thrown from a height of 7 m. After 15 seconds, it reaches its maximum height of 40 m before descending onto the roof of an 18 m high building. The projectile bounces and, 4 seconds later, is at a height of 20 m. The first trajectory follows the model of an absolute value function and the second one follows the model of a square root function whose vertex corresponds to the point where it hits the roof of the building.

The projectile hits the wall of another building at a height of 25 m. How many seconds after the projectile was thrown does it hit the wall of the second building?



$y = -2{,}2\,|x - 15| + 40$; $-2{,}2\,|x - 15| + 40 = 18$; $y = \sqrt{x - 25} + 18$.

The projectile hits the wall of the second building 74 s after it is thrown.